RESULTS ON INVARIANT WHISKERED TORI FOR FIBERED HOLOMORPHIC MAPS AND ON COMPENSATED DOMAINS.

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RESULTS ON INVARIANT WHISKERED TORI FOR FIBERED HOLOMORPHIC MAPS AND ON COMPENSATED DOMAINS.

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A la memoria de Milagros, Came, Casilda, mi Parina, Santos y Juan.

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This work is dedicated to the loving memory of my grandparents. Every day I come to think of the very trying circumstances under which they formed our family. I live in awe of their accomplishments.

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SUMMARY

This thesis is composed of two independent papers:

Construction of whiskered invariant tori for fibered holomorphic maps

In this paper we present a very general theory that includes results on the persistence of quasi-periodic orbits of systems subject to quasi-periodic perturbations. Such quasi-periodic systems appear naturally in many applications where systems are subject to external perturbations of quasi-periodic nature. This is joint work with Rafael de la Llave.

The paper takes up Chapters I - VII of the thesis. We give a brief motivation in Chapter I. In Chapter II we introduce informally the spaces of analytic functions and some number-theoretic and non-degeneracy conditions which are rigorously defined in Chapter III. In Chapter IV we present the Main result of the paper, and present the pseudocode for an Algorithm (Algorithm 1) which efficiently computes the invariant objects that have been introduced in Chapter II. A brief review of the most important references in the literature which are related with our work is in Section 4.5. The proof of the main result, theorem 1, takes up Chapters V-VII. We present an outline for the proof of theorem 1 in Section 4.3.

Regularity properties of the boundary of a domain that imply it is compensated

A Domain $\mathcal{D}$ in a Banach space $(\mathcal{B}, \| \ \|_{\mathcal{B}})$ is called *compensated* if the distance $d_{\mathcal{D}}(x, y)$ on $\mathcal{D}$ defined by the infimum of the lengths of arcs joining points $x, y \in \mathcal{D}$ is comparable to $\|x - y\|_{\mathcal{B}}$. This condition is very important for the function theory in $\mathcal{D}$. This is joint work with Rafael de la Llave.

This paper consists of Chapters VIII- X of the thesis. In Chapter VIII, we introduce and motivate the problem briefly. In Chapter IX, we prove Theorem 2, that shows that certain regularity conditions on the boundary of $\mathcal{D}$ guarantee that $\mathcal{D}$ is compensated when $\mathcal{B}$ is finite dimensional. In Chapter X, we present examples showing that these conditions cannot be relaxed and that they do not necessarily imply compensation for infinite dimensional $\mathcal{B}$.

Part I: Construction of whiskered invariant tori for fibered holomorphic maps

CHAPTER I

INTRODUCTION

The goal of this paper is to present a very general theory of persistence for quasi-periodic orbits of systems subject to quasi-periodic perturbations. We will present results for discrete time systems (maps).

Such quasi-periodic systems appear naturally in many applications where systems are subject to external perturbations of quasi-periodic nature. Notably, astronomical systems are subject to perturbations with several frequencies, corresponding to different planets. Other examples arise in biological systems which are subject to seasonal perturbations, machinery subject to perturbation of several parts vibrating with different frequencies, etc.

This problem, of course, has a very long history in the literature, with the main breakthroughs in the theoretical literature happening in the 50s and 60s (the *KAM theory*). The systematic numerical treatment is more recent. In Subsection 4.5 we have tried to give a brief summary of the main precedents of this work. Certainly, it is quite incomplete as a survey of the field which would require at least a book.

Our approach is based on the so-called *parameterization method*. We derive a functional equation that expresses the fact that an embedding corresponds to an invariant curve. This functional equation is then studied by methods of functional analysis. We note that it will become apparent that the invariance equation will involve adjustment of parameters. These parameters are part of the unknowns of the equation. The role of parameters has been emphasized in the theory of quasi-periodic solutions in [46, 47, 45, 44, 43, 16].

The main technical result is Theorem 1. This is an “a-posteriori" theorem, that is, we assume that there is a sufficiently approximate solution satisfying some non-degeneracy conditions and we conclude that there is a true solution. To make precise the meaning of “approximate" will entail defining norms. There are several definitions (analytic, finitely differentiable, Sobolev) that will apply: We will only consider the analytic case, see Section 3.

The approximate solutions assumed by Theorem 1 can be produced in a variety of ways. For example, if the system we consider is close enough to “ integrable", we can take as approximate solutions the exact solutions in the integrable system. In the quasi- integrable case, we can also obtain improved approximate solutions using formal expansions. Finally, we note that we can take as approximate solutions the results of a numerical computation. To validate the numerical solutions, we do not need to study the algorithm, we only need to verify that they indeed satisfy very approximately the invariance equation and estimate the condition numbers.

The proof of Theorem 1 will consist in showing that an iterative procedure of Newton type (the error of the improved solution is quadratic in the original error) converges. This algorithm has mathematical consequences. It has been known since [36] that the quadratic convergence can overcome small divisors.

The iterative procedure we present also has numerical applications (it lends itself to efficient numerical implementations) since it is not based on transformation theory but rather in applying corrections which ameliorate notably the *curse of dimensionality*: The algorithm presented here (see Algorithm 1) is based on manipulating functions of as many variables as the dimension of the invariant object, whereas the transformation theory requires to deal with functions of as many variables as the dimension of phase space. The book [26] is devoted to the parameterization method and its applications to a new generation of algorithms for computing invariant objects. In an ideal computer with arbitrary precision and unlimited memory, these algorithms could compute arbitrarily close to breakdown. In a real computer they can come extremely close. The a-posteriori format of the theorem allows us to compute with confidence even very close to breakdown. We hope this implementation will be done in a future work.

CHAPTER II

SET UP OF THE PROBLEM AND MOTIVATION OF THE ASSUMPTIONS.

In this Chapter we present informally the formulation of the problem and we motivate the assumptions. For the moment, we will just formulate the equations and describe the geometric properties. We, however, postpone the discussion of analysis issues until after we have discussed spaces and norms for several objects. Of course, the analytic definitions are driven by the geometric formulation which we now develop.

The set up we present is very similar to that in [23, 24, 25], but these papers considered only the *hyperbolic* case. In this paper, see Subsection 2.3.3, we allow also *elliptic* directions. This will require also adding more parameters, see Section 2.4.

2.1 The case of maps.

We now formulate the problem for quasi-periodic maps. The results for maps imply the results for differential equations, by taking sections (taking sections is numerically efficient too). The same geometric ideas apply to differential equations, but they are easier to interpret in the case of diffeomorphisms since they just involve comparing points, not points and vector fields.

Let $\mathbb{T}^d$ be the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$. We will consider families of *skew-product diffeomorphisms over a translation on* $\mathbb{T}^d$, that is, we study maps of the form

$$\begin{aligned} F\ :\ \mathbb{T}^d \times \mathcal{M} \times \mathbb{C}^c &\to \mathbb{T}^d \times \mathcal{M} \\ (\theta, x, \beta) &\mapsto (\ \theta+\omega,\ \ f(\theta, x, \beta)\ ). \end{aligned} \tag{1}$$

Where:

- $0 < d, \;\; 0 \le c \le n$ are integers[1],
- $\mathcal{M}$ is an n-dimensional complex analytic manifold,
- $\omega \in \mathbb{T}^d$ is a fixed frequency, which satisfies some number-theoretic conditions stated in Subsection 3.1.14.
- $f : \mathbb{T}^d \times \mathcal{M} \times \mathbb{C}^c \to \mathcal{M}$ is analytic, and we denote

$$f(\theta, z, \beta) := f_\beta(\theta, z)$$

Skew-product maps (1) have the physical interpretation that there is a large system described by motion with several frequencies (in the variable θ). This large system affects (but is not affected by) the evolution of the variable x. One concrete example is a small satellite in the field of several planets, each moving with its own period.

Since we will be working with analytic regularity, it will be useful to assume that $\mathcal{M}$ is a *complex* manifold of (complex) dimension n. Reference [52] also considers this complex manifold case. If the manifold $\mathcal{M}$ in (1) is real, then we need to consider a complex analytic extension $\widetilde{F}$ of F with the property that $\widetilde{F}$ maps real values into real values.

When working with analytic extensions, we will consider that the angle variable θ takes values on a complex *strip* $\mathbb{T}^d_\rho$ containing $\mathbb{T}^d$:

$$\mathbb{T}^d_\rho := \{ \; \theta \in \mathbb{C}^d / \mathbb{Z}^d, \;\; \theta = (\theta_1, \ldots, \theta_d)^t : \;\; |\mathfrak{Im}\; \theta_i| \;\; \le \;\; \rho, \;\; 1 \le i \le d \; \}$$

[1]Here, c is the dimension of the *parameter space* $\mathbb{C}^c$, and it coincides with the dimension of the *central bundle* of a Whiskered splitting, to be defined in Subsection 2.3.1. We remark that d is possibly strictly smaller than n.

We will denote by T_ω the translation on the strip:

$$T_\omega \ : \ \mathbb{T}^d_\rho \to \mathbb{T}^d_\rho$$
$$\theta \mapsto \theta + \omega.$$

Note that when the quasi-periodic effect is small we have that $f_\beta(\theta, x)$ is almost independent of θ, that is

$$f_\beta(\theta, x) \approx \overline{f_\beta}(x), \qquad \theta \in \mathbb{T}^d_\rho$$

Hence, we will refer to situations where

$$|\ D_1 f_\beta(\theta, x)\ | \ \ll \ 1 \tag{2}$$

as the *perturbative case*. The perturbative case has been considered in [52] when $n = d = 1$. Most of our results will be independent of the perturbative assumption, but if it holds we will obtain sharper results.

We say that $\omega \in \mathbb{T}^d$ is *non-resonant* if

$$\omega \cdot k \ \notin \ \mathbb{Z}\ , \qquad k \in \mathbb{Z}^d \setminus \{0\}$$

When the frequency ω is non-resonant, the map (1) does not admit a periodic orbit. The invariant objects which organize the dynamics generated by (1) are *invariant tori*, which are the graph of a map

$$K : \mathbb{T}^d_\rho \longrightarrow \mathcal{M},$$

i.e. tori of the form

$$\mathfrak{T}_K \ = \ \{\ (\theta,\ K(\theta))\ \mid \theta \in \mathbb{T}^d_\rho\ \} \tag{3}$$

2.2 The invariance equation.

Since a point $(\theta,\ K(\theta))$ maps under (1) to $(\ \theta+\omega,\ f_\beta(\theta, K(\theta)\ )$, we see that the torus $\mathfrak{T}_K$ is invariant if and only if the map K satisfies

$$f_\beta(\theta,\ K(\theta)) = K(\theta+\omega), \qquad \theta \in \mathbb{T}^d_\rho \tag{4}$$

The equation (4) will be the centerpiece of our analysis. We can think of it as a functional analysis problem. As we will see later, obtaining solutions of equation (4) requires that we consider, rather than a single map, a *family* of maps indexed by the parameter β, and that we adjust the parameter. The role of the parameter β in obtaining solutions of the invariance equation (4) was emphasized already in [46], and we explain it informally in Subsection 2.3.5. We remark that the unknowns of the invariance equation (4) are *both* K and β.

We will use the shorthand notation

$$f_\beta \circ K(\theta) := f_\beta(\theta, K(\theta))$$

2.3 The iterative procedure for Euclidean $\mathcal{M}$

This is a case that appears often in applications and it leads to valuable intuition. If $\mathcal{M}$ is an Euclidean manifold, we can write (4) as

$$f_\beta \circ K - K \circ T_\omega = 0 \tag{5}$$

Of course, in a general manifold one cannot compare points by substracting them, but one can use exponential mappings, so that adapting the results of this paper from Euclidean manifolds to general Riemannian manifolds is essentially straightforward and mainly typographical effort (see Section 2.4 for details).

We will try to solve the parameterized equation (5) by a Newton method. We start from an approximate solution $K_0 : \mathbb{T}^d_\rho \to \mathbb{C}^n$, $\beta_0 \in \mathbb{C}^c$ of (5)

$$f_{\beta_0} \circ K_0 - K_0 \circ T_\omega = e_0, \tag{6}$$

where $e_0 : \mathbb{T}^d_\rho \to \mathbb{C}^n$, the error of the approximate solution $(K_0,\ \beta_0)$, will be thought of as small[2]. We try to find $\Delta_0 : \mathbb{T}^d_\rho \to \mathbb{C}^n$, a correction to K_0, and $\delta_0 \in \mathbb{C}^c$, a correction to β_0, in such a way that $(\Delta_0,\ \delta_0)$ eliminate e_0 in first order, that is,

$$D_2 f_{\beta_0} \circ K_0\ \Delta_0 + \partial_\beta f_{\beta_0} \circ K_0\ \delta_0 - \Delta_0 \circ T_\omega = -e_0 \tag{7}$$

If $(\Delta_0,\ \beta_0)$ satisfy (7), then the corrected torus $K_1 := K_0 + \Delta_0$ and parameter $\beta_1 := \beta_0 + \delta_0$ are much improved approximate solutions of (5):

$$f_{\beta_1} \circ K_1 - K_1 \circ T_\omega = \mathcal{E}_2(f)\left[(K_0,\ \beta_0)\,;\ \Delta_0,\ \delta_0\right], \tag{8}$$

where we denoted by

$$\mathcal{E}_2(f)\left[K_0, \beta_0; \Delta_0, \delta_0\right] := f_{\beta_1} \circ K_1 - \left(f_{\beta_0} \circ K_0 + D_2 f_{\beta_0} \circ K_0\ \Delta_0 + \partial_\beta f_{\beta_0} \circ K_0\ \delta_0\right)$$

the error of the linearization of f at $(K_0,\ \beta_0)$, in the direction $(\Delta_0,\ \delta_0)$ (see diagram 2.3). If the correction (Δ_0, δ_0), obtained as a solution of equation (7), is of the same order of magnitude as the error e_0 and if, furthermore, the image of $\mathbb{T}^d_\rho$ under the improved torus K_1 still lies in the domain of analyticity of f then, by Taylor's estimate, the error

$$e_1 := f_{\beta_1} \circ K_1 - K_1 \circ T_\omega$$

will be quadratic with repect to e_0. We remark that equation (7) has two unknowns: We will use an additional equation in order to determine δ_0, see Section 2.4.

Given an analytic map $K : \mathbb{T}^d_\rho \to \mathbb{C}^n$, we will denote

[2] In order to make rigorous sense of the meaning of "small" here, we need to introduce function spaces and norms. We delay this until Chapter 3.

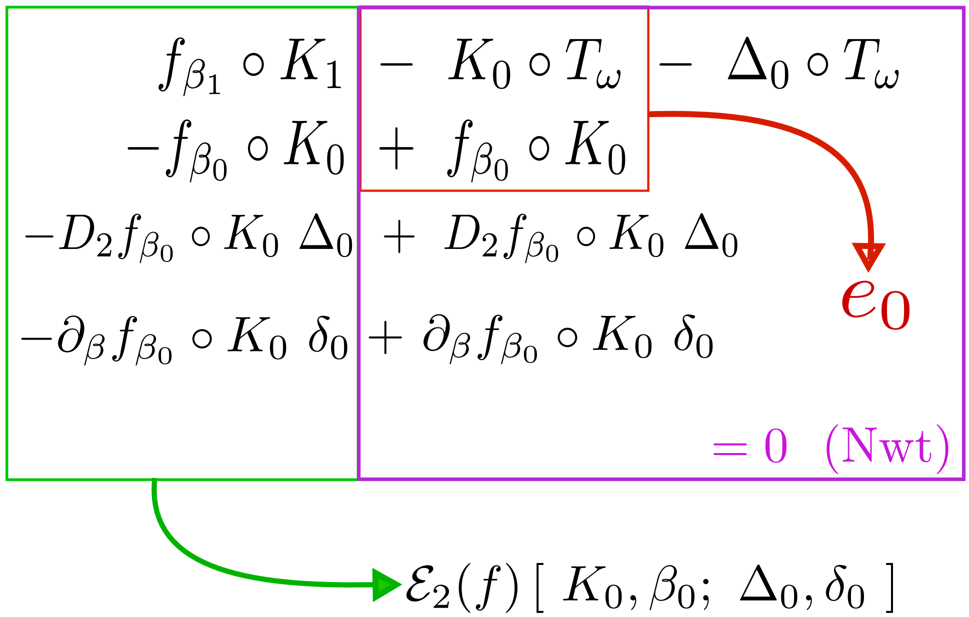

Figure 1: The sum of all terms inside the green and purple boxes is e_1. The Newton equation (7) guarantees that the purple box vanishes. Thus, (8) holds.

$$A_{K,\beta}(\theta) := D_2 f_\beta \circ K(\theta)$$

The *Newton equation* (7) relates the corrections $\Delta_0 \circ T_\omega$ and Δ_0 by an affine transformation whose linear part is A_{K_0,β_0}. Applying (7) recursively, we obtain

$$\Delta_0 \circ T_{n\omega} = A^{(n)}_{K_0,\beta_0} \; \Delta_0 \;\; + \sum_{j=0}^{n-1} A^{(j)}_{K_0,\beta_0} \circ T_{(n-j)\omega} \left(e_0 + \partial_\beta f_{\beta_0} \circ K_0 \; \delta_0\right) \circ T_{(n-j-1)\omega} \;, \tag{9}$$

where we have introduced the products

$$\begin{aligned} A^{(n)}_{K,\beta} &:= A_{K,\beta} \circ T_{(n-1)\omega} \; A_{K,\beta} \circ T_{(n-2)\omega} \dots A_{K,\beta} \\ A^{(-n)}_{K,\beta} &:= A^{-1}_{K,\beta} \circ T_{-(n-1)\omega} \dots \; A^{-1}_{K,\beta} \circ T_{-\omega} \; A^{-1}_{K,\beta} \end{aligned} \tag{10}$$

Note that, for $k, \; l \in \mathbb{Z}$, $A^{(k+l)}_{K,\beta} = A^{(k)}_{K,\beta} \circ T_{l\omega} \; A^{(l)}_{K,\beta}$. It is customary to refer to $A^{(n)}_{K,\beta}$ as *cocycles over a translation*: A *discrete* $GL(n, \mathbb{C})$*-valued cocycle over* T_ω is a map

$$\mathsf{B} : \mathbb{Z} \times \mathbb{T}^d_\rho \longrightarrow GL(n, \mathbb{C})$$

$$B(k, \theta) := B^{(k)}(\theta)$$

such that $B^{(k+l)} = B^{(k)} \circ T_{l\omega} \; B^{(l)}$ holds for all $k, \; l \in \mathbb{Z}$.

2.3.1 Geometric assumptions on $A_{K,\beta}$

It is now clear that we will need to make assumptions on the asymptotic behavior of the products $A^{(\pm n)}_{K,\beta}$: We now introduce informally three assumptions (**H1, H2, H3** below) which we will require for the cocycle A_{K_0,β_0}

H1 There exist analytic vector bundles $E^s_0,\ E^u_0,\ E^c_0$ over $\mathbb{T}^d_\rho$ (denoted, respectively, as the *stable, unstable and central* bundles) such that, for all $\theta \in \mathbb{T}^d_\rho$,

$$E^s_0(\theta) \oplus E^u_0(\theta) \oplus E^c_0(\theta) \simeq \mathbb{C}^n$$

where, for $\sigma \in \{s, u, c\}$, we have denoted by $E^\sigma(\theta)$ the fiber of E^σ over θ. We will say that the bundles $(E^s_0,\ E^c_0,\ E^u_0)$ form a *splitting* E_0 over $\mathbb{T}^d_\rho$. We will furthermore assume that:

- Each of the bundles E^σ_0, $\sigma \in \{s, u, c\}$ is sufficiently approximately invariant with respect to the natural action of A_{K_0,β_0}.

- The splitting E_0 is *whiskered* with respect to A_{K_0,β_0}: For the full definition of whiskered splitting, see Subsection 3.1.12. We now present an informal definition which suffices for the purposes of this introduction.

Informally, we will think of a *whiskered* splitting as an analytic splitting $(E^s_0,\ E^c_0,\ E^u_0)$ such that the standard exponential trichotomy holds[3]: There exist positive real numbers $C_h,\ \mu_s,\ \mu_u,\ \mu_{c,+},\ \mu_{c,-}$ satisfying:

$$\mu_s < 1,\ \mu_u < 1,\quad 1 < \mu_{c,+},\ 1 < \mu_{c,-},\quad \mu_s\mu_{c,+} < 1,\quad \mu_u\mu_{c,-} < 1$$

[3]the existence of an exponential trichotomy has been used very often (see e.g. [58, 18]).

such that

$$v \in E^s \iff \left| A_{K,\beta}^{(n)} v \right| \le C_h \mid v \mid \mu_s^n, \quad n \ge 0$$

$$v \in E^u \iff \left| A_{K,\beta}^{(-n)} v \right| \le C_h \mid v \mid \mu_u^n, \quad n \ge 0$$

$$v \in E^c \iff \begin{cases} \left| A_{K,\beta}^{(n)} \right| \le C_h \mid v \mid \mu_{c,+}^n, & n \ge 0 \\ \\ \left| A_{K,\beta}^{(-n)} \right| \le C_h \mid v \mid \mu_{c,-}^n, & n \ge 0 \end{cases}$$

We will need to modify this preliminary definition of Whiskered splitting (which is too strong for *approximately* invariant splittings) by considering projected cocycles (see Subsection 3.1.12).

2.3.2 Solution of the Newton equation (7) in the hyperbolic directions

The Newton equation (7) is equivalent to a system of linear equations

$$A_{K_0,\beta_0} \, \Delta_0^\sigma \; - \; \Delta_0^\sigma \circ T_\omega \; = \; -e_0^\sigma - \partial_\beta f_{\beta_0} \circ K_0 \; \delta_0^\sigma \;, \quad \sigma \in \{s, u, c\} \tag{11}$$

where

- The unknown Δ_0^σ is a section of the bundle E_0^σ.
- $e_0^\sigma = e_0^{\sigma,\sigma} + \tilde{e}_0^\sigma$, where $e_0^{\sigma,\sigma}$ is the projection of e_0 onto E_0^σ, and where the *non-diagonal* error $\tilde{e}_0^\sigma$ is zero if E_0^σ is invariant under A_{K_0,β_0}.

Note that we have set the unknown δ_0^σ (related to the correction of the parameter) in the right-hand side of (11): We explain briefly why we choose to present the system (11) in this way in Subsection 2.3.3. Assuming a solution of the system (11) is found, we then set

$$\Delta_0 := \Delta_0^s + \Delta_0^u + \Delta_0^c, \qquad \delta_0 := \delta_0^s + \delta_0^u + \delta_0^c$$

The Whitney sum $E_0^s \oplus E_0^u$ is informally called the *hyperbolic directions* of the splitting. It has been noted in [20] that the system (11), for $\sigma \in \{s, u\}$, can be solved via a geometric iteration, which we implement in Section 5.1. Since the method of 5.1 does not require to work in coordinates, we will not need to assume that the bundles E_0^s, E_0^u are trivilizable.

2.3.3 Elliptic directions: Approximate Reducibility and first Melnikov condition

The Newton equation (7) along the center directions will be dealt with by non-geometric methods such as Fourier series, which require taking coordinates. The second geometric condition guarantees that this can be done:

H2 The central bundle E_0^c is *trivializable over* $\mathbb{T}^d_\rho$ via a vector bundle isomorphism

$$\psi_0 : \mathbb{T}^d_\rho \times \mathbb{C}^c \longrightarrow E_0^c$$

Let $\Pi^c_{E_0}$ be the projection onto E_0^c, and let

$$A^{c,c}_{K_0,\beta_0} := \Pi^c_{E_0} \circ T_\omega \; A_{K_0,\beta_0} \; \Pi^c_{E_0}$$

Using ψ_0, we can find a matrix representation of $A^{c,c}_{K,\beta}$ as follows:

$$\overline{A^{c,c}_{K_0,\beta_0}} := \psi_0^{-1} \circ T_\omega \; A^{c,c}_{K_0,\beta_0} \; \psi_0$$

Assumption **H3** appears already in [46], but [46] assumes that it holds in all the linearization and not only along the central directions.

H3 For fixed $\lambda_1, \ldots \lambda_c \in \mathbb{C}$ which satisfy some number-theoretic conditions (stated in Subsection 3.1.14), there exists parameter $\beta_0 \in \mathbb{C}^c$ and an analytic bundle of frames

$$U_0 : \mathbb{T}^d_\rho \longrightarrow GL(c, \mathbb{C})$$

in such a way that

$$U_0^{-1} \circ T_\omega \ \ \overline{A^{c,c}_{K_0,\beta_0}} \ \ U_0 - \Lambda = e_{\mathscr{R}} \tag{12}$$

where $\Lambda = \mathrm{diag}[\lambda_1, \ldots, \lambda_c]$, and $e_{\mathscr{R}} \in GL(c, \mathbb{C})$ is sufficiently small.

If (12) holds with $e_{\mathscr{R}} \equiv 0$, it is said that $\overline{A^{c,c}_{K_0,\beta_0}}$ is *reducible* to Λ. If (12) holds, we will say that $\overline{A^{c,c}_{K_0,\beta_0}}$ is *approximately reducible.*

Remark 1. *The geometric meaning of* (12) *is that we can find a linear change of variables in such a way that the coordinate expression of the linearized equations* $\overline{A^{c,c}_{K_0,\beta_0}}$ *is expressed as multiplication by a constant matrix* Λ*, and that* Λ *remains fixed throughout the iterative procedure. This is a strong assumption, and one cannot expect it to hold for a fixed map. However, we will show (see Subsection 2.3.5) that it can hold throughout the iterative procedure if* f *is embedded in a family* $\{f_\beta\}_{\beta \in \mathbb{C}^c}$ *depending on a c-dimensional parameter* β *and if, furthermore, a non-degeneracy condition (see hypothesis* ***[H4]*** *below) is satisfied.*

Using the trivialization ψ_0 and the approximate reducibility identity (12) we see that the Newton equation (11) for $\sigma = c$ is equivalent to:

$$\Lambda \ \overline{\Delta_0^c} \ - \ \overline{\Delta_0^c} \circ T_\omega \ = \ U_0^{-1} \circ T_\omega \ [\, -e_0^c \ - \ \partial_\beta f_{\beta_0} \circ K_0 \ \delta_0^c \,] - e_{\mathscr{R}} U_0^{-1} \overline{\Delta_0^c} \tag{13}$$

where $\overline{\Delta_0^c} := U_0^{-1} \ \psi_0^{-1} \ \Delta_0^c$. It is standard in KAM theory to argue that the term $e_{\mathscr{R}} U_0^{-1} \overline{\Delta_0^c}$ is of quadratic order and hence deleting it will not affect the quadratic convergence of the Newton iteration (we will indeed show that this is the case).

The resulting equation:

$$\Lambda \ \overline{\Delta_0^c} \ - \ \overline{\Delta_0^c} \circ T_\omega \ = \ U_0^{-1} \circ T_\omega \ [\, -e_0^c \ - \ \partial_\beta f_{\beta_0} \circ K_0 \ \delta_0^c \,] \tag{14}$$

is a constant coefficients *cohomology equation.* Equating Fourier coefficients of ω, we readily obtain a formal solution (see equation (47) in Section 5.5). A small divisor problem, involving both the spectrum of Λ and the frequency ω, is present in the formal solution. To overcome the small divisors, we will require (as is standard in the theory of lower dimensional elliptic tori: See Section 4.5) that Λ and ω jointly satisfy a number theoretic condition called the *first Melnikov* condition (see Subsection 3.1.14).

2.3.4 Linear equations for the invariance of the splitting

The system of linear equations (11) involves non-diagonal errors because the splitting $(E_0^s, \; E_0^c, \; E_0^u)$ is only approximately invariant. Hence, at each step of the iterative procedure, we must also construct a more approximately invariant splitting from $(E_0^s, \; E_0^c, \; E_0^u)$, and reformulate system (11) with respect to the improved splitting.

We thus need to complement system (11) with a Newton equation for the improvement of the splitting. To this end, we introduce the well known coordinates of the Grassmannian (modelled after a space of linear transformations) in Subsection 3.1.4. Using the coordinates in Subsection 3.1.4, a neighborhood of the splitting $(E_0^s, \; E_0^c, \; E_0^u)$ is parameterized by an open set of a Banach space (see 3.1.5). This, in turn, allows to introduce functional equations for the invariance of the splitting in Subsection 4.1.2. Geometrically speaking, a more approximate solution of the functional equations in 4.1.2 correspond to a more approximately invariant splitting (see Proposition 1).

The additive corrections for the splitting $(E_0^s, \; E_0^c, \; E_0^u)$ will be obtained as the solutions of the Newton equations associated to the functionals presented in Subsection 4.1.2. The Newton equations are found in Section 6.2. We solve the Newton equations for the invariance of the splitting using a geometric iteration (see Section 5.4) similar to that used for the Newton equation (11) along the hyperbolic directions.

2.3.5 Linear equation for the reducibility: The role of parameters, second Melnikov condition and a non-degeneracy condition

We now observe that the trivialized Newton equation for the invariance of the torus along the central directions (13) involves the error $e_{\mathscr{R}}$ of the approximate reducibility identity (12). Hence, at each step of the iterative procedure, we must also compute a frame U' under which $\overline{A^{c,c}_{K,\beta}}$ is more approximately reducible. At each step, we will reformulate equation (13) with the improved frame U'.

We implement a quasi-Newton method for U: Let W be the additive correction (i.e. we will set $U' = U + W$). Proceeding as we did to obtain equation (14), we obtain the following cohomology equation for W

$$\Lambda\, W \ - \ W \circ T_\omega\ \Lambda \ = \ -\, U^{-1} \circ T_\omega\ e_{\mathscr{R}} - \eta_{\mathscr{R}}\ [\delta] \tag{15}$$

where $\eta_{\mathscr{R}}$ is an affine function[4] which appears because of the linear terms in the linearization of $\overline{A^{c,c}_{K,\beta}}$. See Proposition 2 for the expression of $\eta_{\mathscr{R}}$.

Using the Fourier method, we obtain a formal solution (see equation (75) in Section 5.5). A new small divisor problem appears, now involving also the differences of points in the spectrum of Λ. To overcome these small divisors, we will require that the pair $(\Lambda,\ \omega)$ satisfy the *second Melnikov condition* (see Subsection 3.1.14).

Remark 2. *It is not strictly necessary to impose the second Melnikov condition on the pair* $(\Lambda,\ \omega)$ *in order to just compute an invariant torus near* $\mathfrak{K}_0$*: An* almost-reducibility *scheme can be implemented (see [14], [10]). However, the tori produced by an almost-reducibility scheme may fail to be reducible (see [15]).*

An additional obstruction arises for the diagonal terms of equation (15): We must

[4]For an affine function η, we will denote by $\eta[\delta]$ the evaluation of η at δ.

ensure that

$$\left[\, U^{-1} \circ T_\omega \; e_{\mathscr{R}} - \eta_{\mathscr{R}}[\delta] \,\right]\widehat{}_{jj}(0) \;=\; 0 \;,\; 1 \leq j \leq c \tag{16}$$

and that (16) is satisfied throughout the Newton iteration. If the *non - degeneracy* condition

H4 The linear part of the affine function $\eta_{\mathscr{R}}$ is an isomorphism.

is satisfied, then there exists a unique δ such that (16) holds. Thus, the parameter correction δ is determined by the linear equation (16).

Note that **[H1], [H2], [H3], [H4]** are open conditions and, assuming that certain condition numbers are satisfied for the initial data of the iterative procedure, we can ensure that **[H1], [H2], [H3], [H4]** are satisfied at each step.

2.4 Extension to general manifolds.

The above considerations can be formulated for $\mathcal{M}$ a Riemannian manifold using standard tools: Exponential mappings and connectors [28]. The error in the invariance equations can be measured by

$$e \;=\; \exp^{-1}_{K\circ T_\omega} \; f \circ K$$

We can introduce the correction Δ as a section of $T\mathcal{M}$ restricted to the image of K.

$$\widetilde{K} \;=\; \exp_{K\circ T_\omega} \; \Delta$$

The analogue of the Newton equation (7) is

$$S_{f\circ K}^{K\circ T_\omega}\ D_2 f\circ K - \Delta\circ T_\omega \ = \ -e \tag{17}$$

where S_x^y is the *connector*, i.e. an identification between $T_x\mathcal{M}$ and $T_y\mathcal{M}$ (when x and y are close). We can take:

$$S_x^y = (D\exp_x)\exp_x^{-1} y$$

and we think of S_x^y as parallel transport from $T_x\mathcal{M}$ to $T_y\mathcal{M}$ along the closest geodesic joining x and y. Then, the linearized equations read again:

$$A\ \Delta - \Delta\circ T_\omega \ = \ -e$$

but $A : T_K\mathcal{M} \longrightarrow T_{K\circ T_\omega}\mathcal{M}$ is defined by

$$S_{f\circ K}^{K\circ T_\omega}\ D_2 f\circ K$$

Of course, this is completely consistent with our treatment of the Euclidean manifolds. In that case all the tangent spaces can be identified and the connectors are the identity.

We have now informally introduced all the basic objects involved in the proof of our main result, Theorem 1. An outline of the proof of Theorem 1 is given in Section 4.3, where we emphasize some formal aspects of the iterative procedure. A pseudocode for the iterative procedure is given in Algorithm 1.

CHAPTER III

FUNCTION SPACES, NUMBER-THEORETICAL CONDITIONS, AND OTHER PRELIMINARIES.

In this Chapter we introduce the function spaces and the norms that will be used to measure how approximately are the Invariance and Reducibility equations satisfied. We introduce the number theoretic (Melnikov) conditions that play a role in the proof of Theorem 1. We also fix several notations.

Remark 3. *In this work there will appear many constants which depend only on quantities that are fixed in the statement of Theorem 1. Following standard practice in order to avoid an abundance of such constants, we set C to be a generic constant which does not depend on* `k`*, the step of the Nash-Moser iteration in Algortithm 1. The optimal value of C may change from line to line. For simple model problems, careful bookkeeping of these constants appear e.g. in [8], [19] and are used to obtain effective estimates.*

3.1 Function Spaces that we will use.

3.1.1 Domains

Given $\rho > 0$, we define the *complex strip*:

$$\mathbb{T}^d_\rho := \left\{ \ \theta \in \mathbb{C}^d/\mathbb{Z}^d, \ \theta = (\theta_1, \ldots, \theta_d)^\intercal : |\ \mathfrak{Im}(\theta_j)\ | \ \le \ \rho \ \ , \forall j \ \right\}$$

Let $\eta > 0, \ \rho > 0$, and $\beta \in \mathbb{C}^k$. We denote by $B^k_\eta(\ \beta\ ) \subset \mathbb{C}^k$ the open ball of radius η centered at β. Given a function $K : \mathbb{T}^d_\rho \to \mathbb{C}^n$, we define the following domains:

$$\mathcal{D}_{\rho,\eta}(K, \ \beta) := \mathbb{T}^d_\rho \times \mathcal{D}_\eta(K) \times B^c_\eta(\beta) \subset \mathbb{T}^d_\rho \times \mathbb{C}^n \times \mathbb{C}^c$$

where $\mathcal{D}_\eta(K)$ is the domain

$$\mathcal{D}_\eta(K) := \bigcup_{z \in K\left(\mathbb{T}^d_\rho\right)} B^n_\eta(z)$$

3.1.2 Function spaces.

Let the domain Υ be either $\mathbb{T}^d_\rho$ or $\mathcal{D}_{\rho,\eta}(K,\ \beta)$. Given a complex manifold X, let

$$\mathcal{A}^X_\Upsilon \ := \ \{ \ f: \ \Upsilon \to X \ : \ f \text{ is analytic on } \ \mathrm{Int}\,(\ \Upsilon\ ), \text{ and continuous on } \Upsilon \ \}$$

The spaces $A^X_{\mathbb{T}^d_\rho}$ will appear often in this paper, and we will use the shorthand notation

$$A^X_{\mathbb{T}^d_\rho} \ := \ \mathcal{A}^X_\rho$$

When $(X,\ |\ \cdot\ |_X)$ is a Banach space, $\mathcal{A}^X_\rho$ is a Banach space [1] with the norm

$$|\ g\ |_\rho \ := \ \sup_{\theta \in \mathbb{T}^d_\rho} \ |\ g(\theta)\ |$$

Any $g \in \mathcal{A}^{\mathbb{C}^c}_\rho$ admits a Fourier series:

$$g(\theta) = \sum_{k \in \mathbb{Z}^d} \widehat{g}(k) \cdot \exp(2\pi i k \cdot \theta), \quad \theta \in \mathbb{T}^d_\rho$$

where $\widehat{g}(k)$ is the k-th *Fourier coefficient*

$$\widehat{g}(k) \ := \ \int_{\mathbb{T}^d} g(\theta) \ \exp(-2\pi i \ k \cdot \theta)$$

3.1.3 Some elementary estimates on analytic functions.

We now recall two basic estimates for $g \in \mathcal{A}^{\mathbb{C}^c}_\rho$ (see [59]): The **Cauchy- Fourier estimate** is:

$$|\ \widehat{g}\ (\ k\ )\ | \ \leq \ C \exp(\ -2\pi\rho\ |\ k\ |\ ) \ \ |\ g\ |_\rho \tag{18}$$

The **Cauchy estimate for the derivative** is:

$$|\ Dg\ |_{\rho-\zeta} \ \leq \ C\ \zeta^{-1}\ |\ g\ |_\rho \tag{19}$$

[1] In this paper we will only consider some specific spaces X, which we introduce below.

3.1.4 The Grassmannian G_n^k.

Given $k \leq n$, let G_n^k be the Grassmann manifold of k-dimensional complex subspaces of $\mathbb{C}^n$. The manifold G_n^k is metrizable with the metric

$$d_{G_n^k}(V_1,\ V_2) := \sup_{v\in V_1,\ |v|=1} \inf\{\ |v-w|\ :\ w \in V_2\ \}$$

Let $V \in G_n^k$, and $V^\perp \in G_n^{n-k}$, a complementary subspace (i.e. $V \oplus V^\perp = \mathbb{C}^n$). We introduce the subset of G_n^k defined by:

$$\mathcal{U}_{G_n^k}^{V^\perp}(V)\ :=\ \left\{\ W \in G_n^k \mid W \cap V^\perp = \{\ 0\ \}\ \right\}$$

The set $\mathcal{U}_{G_n^k}^{V^\perp}(V)$ is a neighborhood of $V \subset G_n^k$. It is parameterized by the vector space:

$$\mathcal{L}\left(\ V,\ V^\perp\ \right) := \left\{\ \text{linear maps}\ \ V \to V^\perp\ \right\}$$

The parameterization is as follows: Given $S \in \mathcal{L}(V, V^\perp)$, it is easy to check that there exists a unique $V_S \in \mathcal{U}_{G_n^k}^{V^\perp}(V)$ such that

$$V_S := \{v + Sv\ ,\quad v \in V\}$$

A chart for $\mathcal{U}_{G_n^k}^{V^\perp}(V)$ is

$$\begin{aligned} \Phi_{G_n^k}^{V,V^\perp}\ :\ \mathcal{U}_{G_n^k}^{V^\perp}(V) &\to \mathcal{L}(V, V^\perp) \\ V_S &\mapsto S. \end{aligned}$$

We will not use the well-known fact that the charts $\Phi_{G_n^k}^{V,V^\perp}$, $V \in G_n^k$, endow G_n^k with a complex analytic manifold structure. See [38] for details.

We will denote by $|\cdot|_{\mathcal{L}(V,V^\perp)}$ the operator norm on $\mathcal{L}(V, V^\perp)$. The neighborhood $\mathcal{U}_{G_n^k}^{V^\perp}(V)$ is also metrizable with the metric

$$d_{\mathcal{U}_{G_n^k}^{V^\perp}(V)}(V_{S_1}, V_{S_2}) := |S_1 - S_2|_{\mathcal{L}(V,V^\perp)}$$

Remark 4. *Given $r > 0$, let $\mathcal{U}_{G_n^k}^{V^\perp,\ r}(V)$ be the following neighborhood of V in G_n^k:*

$$\mathcal{U}_{G_n^k}^{V^\perp,\ r}(V) \ := \ \left\{ \ W \in \mathcal{U}_{G_n^k}^{V^\perp}(V) \quad | \quad d_{\mathcal{U}_{G_n^k}^{V^\perp}(V)}(V, W) < r \ \right\}$$

Note that the restrictions of $d_{\mathcal{U}_{G_n^k}^{V^\perp}(V)}$ and $d_{G_n^k}$ on $\mathcal{U}_{G_n^k}^{V^\perp,\ r}(V)$ are equivalent metrics on $\mathcal{U}_{G_n^k}^{V^\perp,\ r}(V)$.

3.1.5 Analytic vector bundles over $\mathbb{T}_\rho^d$.

Let $\mathcal{V}_\rho^k$ be the set of *analytic vector bundles of rank k over* $\mathbb{T}_\rho^d$:

$$\pi_{E^\sigma} \ : \ E \longrightarrow \mathbb{T}_\rho^d$$

$$\left\{ \ \pi_E^{-1}(\theta) \ \right\} \ := \ E(\theta) \ \in G_n^k$$

and such that $E \in \mathcal{A}_\rho^{G_n^k}$, where we have denoted also by E the function

$$\begin{aligned} E \ : \ \mathbb{T}_\rho^d &\to G_n^k \\ \theta &\mapsto E(\theta). \end{aligned}$$

We endow $\mathcal{V}_\rho^k$ with the metric

$$d_{\mathcal{V}_\rho^k}(E_1, \ E_2) \ := \ \sup_{\theta \in \mathbb{T}_\rho^d} \ d_{G_n^k} \left(\ E_1(\theta), \ E_2(\theta) \ \right)$$

Given $\alpha \in \mathbb{T}^d$, we introduce the vector space of analytic sections over the translation T_α and onto the vector bundle E^σ:

$$\mathcal{A}_{\rho,\alpha}^{E^\sigma} \ := \ \left\{ \ f \in \mathcal{A}_\rho^{\mathbb{C}^n} \ | \ f(\theta) \in E^\sigma(\theta + \alpha) \ , \ \theta \in \mathbb{T}_\rho^d \ \right\}$$

with the norm $|f|_\rho := \sup_{\theta \in \mathbb{T}_\rho^d} |\ f(\theta)\ |$, $\mathcal{A}_{\rho,\ \alpha}^{E^\sigma}$ is a Banach space.

3.1.6 Analytic Vector bundle homomorphisms over a translation T_α on $\mathbb{T}_\rho^d$

Given $E_1 \in \mathcal{V}_\rho^{k_1}$, $E_2 \in \mathcal{V}_\rho^{k_2}$, $\alpha \in \mathbb{T}^d$ we denote by $\mathcal{A}_{\rho,\alpha}^{\mathcal{L}(E_1,\ E_2)}$ the vector space of analytic vector bundle homomorphisms $E_1 \to E_2$ over T_α: It is the set of analytic functions

$$A : E_1 \longrightarrow E_2$$

such that, for any $\theta \in \mathbb{T}^d_\rho$:

1. The image of the fiber $E_1(\theta)$ under A is contained in the fiber $E_2(\theta+\alpha)$.

2. Given $\theta \in \mathbb{T}^d_\rho$, let $A(\theta) \;:=\; A|_{E_1(\theta)}$. Then,

$$A(\theta) \;\in\; \mathcal{L}\left(E_1(\theta),\; E_2(\theta+\alpha)\right).$$

Let $E \in \mathcal{V}^j_\rho$. We denote by $\mathcal{A}^{GL(E)}_{\rho,\alpha}$ the vector space of analytic vector bundle isomorphisms $E \to E$ over T_α:

$$\mathcal{A}^{G\mathcal{L}(E)}_{\rho,\alpha} \;:=\; \left\{\; A \in \mathcal{A}^{\mathcal{L}(E,\; E)}_{\rho,\alpha} \mid A(\theta) \in GL\left(E(\theta),\; E(\theta+\alpha)\right)\;,\; \theta \in \mathbb{T}^d_\rho \;\right\}$$

With the norm $\mid A \mid_\rho := \sup_{\theta\in\mathbb{T}^d_\rho} \mid A(\theta) \mid_{\mathcal{L}(E(\theta),\; E(\theta+\alpha))}$, $\mathcal{A}^{\mathcal{L}(E,\; E)}_{\rho,\alpha}$ is a Banach space. We will also use the following Banach algebra type inequality: Let $\alpha,\; \beta \in \mathbb{T}^d$, $E_1 \in \mathcal{V}^{k_1}_\rho,\; E_2 \in \mathcal{V}^{k_2}_\rho,\; E_3 \in \mathcal{V}^{k_3}_\rho$ $A \in \mathcal{A}^{\mathcal{L}(E_1,\; E_2)}_{\rho,\alpha}$, $B \in \mathcal{A}^{\mathcal{L}(E_2,\; E_3)}_{\rho,\beta}$. Then, $A\; B \in \mathcal{A}^{\mathcal{L}(E_1, E_3)}_{\rho,\alpha+\beta}$ and

$$\mid A\; B \mid_\rho \;\leq\; \mid A \mid_\rho \;\mid B \mid_\rho$$

3.1.7 Analytic splittings over $\mathbb{T}^d_\rho$

We will assume that $E_0 \in \mathcal{A}^{G^s_n\times G^u_n\times G^c_n}_\rho$, an *analytic splitting over* $\mathbb{T}^d_\rho$, is given as data for the iterative procedure (recall the discussion in Subsection 2.3.4). The splitting E_0 consists of three vector bundles $E^\sigma_0 \in \mathcal{V}^\sigma_\rho$, $\sigma \in \{s,\; u,\; c\}$ [2] such that for all $\theta \in \mathbb{T}^d_\rho$, it holds that

$$E^s_0(\theta) \;\oplus\; E^u_0(\theta) \;\oplus\; E^c_0(\theta) = \mathbb{C}^n$$

[2] We use the letter σ to denote both the vector bundles (s for "stable", u for "unstable", c for "central") and their rank. Note that $s+u+c=n$.

3.1.8 Charts for $\mathcal{V}_\rho^k$

Let $\{\sigma, \sigma', \sigma''\}$ be a permutation of the set $\{s, u, c\}$. We denote the *Whitney sum* of the vector bundles $E_0^{\sigma'}$ and $E_0^{\sigma''}$ by

$$E_0^{\sigma' \oplus \sigma''} \ := \ E_0^{\sigma'} \ \oplus \ E_0^{\sigma''} \in \mathcal{V}_\rho^{\sigma' + \sigma''}$$

Note that the bundles $E_0^{\sigma' \oplus \sigma''}$ and E_0^σ are complementary:

$$E_0^{\sigma' \oplus \sigma''}(\theta) \ \oplus E_0^\sigma(\theta) = \mathbb{C}^n, \quad \theta \in \mathbb{T}_\rho^d$$

We introduce the set

$$\mathcal{U}_{\mathcal{V}_\rho^\sigma}(E_0^\sigma) := \left\{ \ E^\sigma \in \mathcal{V}_\rho^\sigma \mid E^\sigma(\theta) \cap E_0^{\sigma' \oplus \sigma''}(\theta) = \{ \ 0 \ \}, \ \theta \in \mathbb{T}_\rho^d \ \right\}$$

Given $S^\sigma \in \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma,\ E_0^{\sigma' \oplus \sigma''}\right)}$, there exists a unique $E_S^\sigma \in \mathcal{U}_{\mathcal{V}_\rho^\sigma}(E_0^\sigma)$ such that

$$E_S^\sigma(\theta) = \{ \ v + S^\sigma(\theta) \ v \mid v \in E_0^\sigma(\theta) \ \} \ , \quad \theta \in \mathbb{T}_\rho^d$$

Since $\mathbb{T}_\rho^d$ is compact, $\mathcal{U}_{\mathcal{V}_\rho^\sigma}(E_0^\sigma)$ is a neighborhood of $E_0^\sigma \in \mathcal{V}_\rho^\sigma$. It is parameterized via the chart

$$\begin{aligned} \Phi_{\mathcal{V}_\rho^\sigma}^{E_0^\sigma, E_0^{\sigma' \oplus \sigma''}} \ : \ \mathcal{U}_{\mathcal{V}_\rho^\sigma}(E^\sigma) &\to \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma,\ E_0^{\sigma' \oplus \sigma''}\right)} \\ E_S^\sigma &\mapsto S^\sigma. \end{aligned}$$

The neighborhood $\mathcal{U}_{\mathcal{V}_\rho^\sigma}(E_0^\sigma)$ is also metrizable with the metric

$$d_{\mathcal{U}_{\mathcal{V}_\rho^\sigma}(E_0^\sigma)}(E_{S_1}^\sigma, E_{S_2}^\sigma) \ := \ | \ S_1 - S_2 \ |_\rho$$

Remark 5. *Given $r > 0$, let $\mathcal{U}_{\mathcal{V}_\rho^\sigma}^r(E_0^\sigma)$ be the following neighborhood of E_0^σ in $\mathcal{V}_\rho^\sigma$:*

$$\mathcal{U}_{\mathcal{V}_\rho^\sigma}^r(E_0^\sigma) \ := \ \left\{ \ E_S^\sigma \in \mathcal{U}_{\mathcal{V}_\rho^\sigma}(E_0^\sigma) \quad | \quad d_{\mathcal{U}_{G_n^k}^{V^\perp}(V)}(E_S^\sigma,\ E_0^\sigma) < r \ \right\}$$

Note that the restrictions of $d_{\mathcal{U}_{G_n^k}^{V^\perp}(V)}(E_S^\sigma,\ E_0^\sigma)$ and $d_{\mathcal{V}_\rho^\sigma}$ on $\mathcal{U}_{\mathcal{V}_\rho^\sigma}^r$ are equivalent metrics on $\mathcal{U}_{\mathcal{V}_\rho^\sigma}^r$.

Remark 6. *Given three vector bundle homomorphisms*

$$S^\sigma \in \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma,\ E_0^{\sigma'\oplus\sigma''}\right)} \ , \quad \sigma \in \{s,u,c\},$$

we will denote by E_S the splitting $(E_S^s,\ E_S^u,\ E_S^c)$.

Remark 7. *The initial data $E_0 \in \mathcal{A}_\rho^{G_n^s \times G_n^u \times G_n^c}$ is a geometric object which is not well suited for the operations that will be performed in Algorithm 1. In particular, $\mathcal{A}_\rho^{G_n^s \times G_n^u \times G_n^c}$ is not a vector space, so it does not make sense to perform additive corrections as will be done at each step of the Nash Moser iteration.*

To overcome this difficulty, we will apply the chart $\Phi_{\mathcal{V}_\rho^\sigma}^{E_0^\sigma, E_0^{\sigma'\oplus\sigma''}}$ to the splitting E_0, which is given as data for the first step of the iterative procedure. Note that

$$\Phi_{\mathcal{V}_\rho^\sigma}^{E_0^\sigma, E_0^{\sigma'\oplus\sigma''}}(E_0^\sigma) = 0 \ , \quad \sigma \in \{s,c,u\}$$

Then we will work for the rest of the Nash-Moser iteration in the vector spaces $\mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma,\ E_0^{\sigma'\oplus\sigma''}\right)}$, where it makes sense to solve linearized equations and perform additive corrections. Hence, at each step of the iterative procedure, we are given as data three vector bundle linear homomorphisms

$$S^\sigma \in \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma,\ E_0^{\sigma'\oplus\sigma''}\right)} \ , \quad \sigma \in \{s,u,c\}$$

we will then compute additive corrections [3]

$$\chi^\sigma \in \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma,\ E_0^{\sigma'\oplus\sigma''}\right)} \ , \quad \sigma \in \{s,u,c\}$$

such that the bundles

$$E_{S+\chi}^\sigma := \left(\Phi_{\mathcal{V}_\rho^\sigma}^{E_0^\sigma, E_0^{\sigma'\oplus\sigma''}} \right)^{-1} (S^\sigma + \chi^\sigma) \ , \quad \sigma \in \{s,u,c\}$$

---

[3] *The corrections χ^σ are obtained as solutions of linear equations, see Lemma 4, for which the fact that $\mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma,\ E_0^{\sigma'\oplus\sigma''}\right)}$ is a vector space is also used.*

are more approximately invariant under the action of the cocycle $A_{K,\beta}$ (recall the discussion in Subsection 2.3.1).

The corrections $S^\sigma + \chi^\sigma$ will be improved solutions of a functional equation, introduced in Subsection 4.1.2, which expresses the invariance of the splitting. The invariance equation for the splitting in Subsection 4.1.2 admits a geometric interpretation, which we present in Proposition 1.

3.1.9 Projections associated to E_S.

The splitting E_S defines a bundle of projections onto each E_S^σ ($\sigma \in \{s, u, c\}$):

$$\Pi_{E_S}^\sigma \ := \ \Pi_S^\sigma \ \in \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(\mathbb{C}^n \times \mathbb{T}_\rho^d, \mathbb{C}^n \times \mathbb{T}_\rho^d\ \right)}$$

Π_S^σ is characterized as follows : Given $v \in \mathbb{C}^n(\theta)$ let

$$\Pi_S^\sigma(\theta)\ v \ := \begin{cases} 0 \text{ if } v \in E_S^{\sigma' \oplus \sigma''}(\theta) \\ \\ v \text{ if } v \in E_S^\sigma(\theta) \end{cases}$$

and extend linearly. We also introduce the bundle of projections $\Pi_S^{\sigma' \oplus \sigma''}$, which is characterized as follows: Given $v \in \mathbb{C}^n(\theta)$ let

$$\Pi_S^{\sigma' \oplus \sigma''}(\theta)\ v \ := \begin{cases} 0 \text{ if } v \in E_S^\sigma(\theta) \\ \\ v \text{ if } v \in E_S^{\sigma' \oplus \sigma''}(\theta) \end{cases}$$

3.1.10 Coordinates of a Whitney sum of bundles

In this Subsection, given $\sigma,\ \sigma' \in \{s, u, c\}$ with $\sigma \neq \sigma'$ and two vector bundle homomorphisms

$$S^\sigma \in \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma,\ E_0^{\sigma' \oplus \sigma''}\right)} \ , \quad S^{\sigma'} \in \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^{\sigma'},\ E_0^{\sigma \oplus \sigma''}\right)}$$

we compute $\Phi_{\mathcal{V}_\rho^{\sigma+\sigma'}}^{E_0^\sigma \oplus E_0^{\sigma'}, E_0^{\sigma''}}(E_S^\sigma \oplus E_S^{\sigma'})$, the coordinate corresponding to the Whitney sum of the bundles E_S^σ, $E_S^{\sigma'}$ under the chart $\Phi_{\mathcal{V}_\rho^{\sigma+\sigma'}}^{E_0^\sigma \oplus E_0^{\sigma'}, E_0^{\sigma''}}$.

We remark that the coordinate $\Phi^{E_0^\sigma \oplus E_0^{\sigma'}, E_0^{\sigma''}}_{\mathcal{V}_\rho^{\sigma+\sigma'}}(E_S^\sigma \oplus E_S^{\sigma'})$ is not just $S^\sigma \oplus S^{\sigma'}$ since the range of $S^\sigma \oplus S^{\sigma'}$ has a non-trivial projection on $E_0^\sigma \oplus E_0^{\sigma'}$.

To find the correct expression for $\Phi^{E_0^\sigma \oplus E_0^{\sigma'}, E_0^{\sigma''}}_{\mathcal{V}_\rho^{\sigma+\sigma'}}(E_S^\sigma \oplus E_S^{\sigma'})$, note that $E_S^\sigma \oplus E_S^{\sigma'}$ is parameterized by $E_0^\sigma \oplus E_0^{\sigma'}$ as follows:

$$\begin{aligned}\Big(E_S^\sigma \oplus E_S^{\sigma'} \Big)(\theta) &= v + w + S^\sigma(\theta)v + S^{\sigma'}(\theta)w\ ,\ v \in E_0^\sigma(\theta),\ w \in E_0^{\sigma'}(\theta) \\ &= \underbrace{\Big(v + S^{\sigma',\sigma}(\theta)w\Big)}_{\text{call this } v_1} + \underbrace{\Big(S^{\sigma,\sigma'}(\theta)v + w\Big)}_{\text{call this } v_2} + \underbrace{S^{\sigma,\sigma''}(\theta)v + S^{\sigma',\sigma''}(\theta)w}_{\text{call this } v_3}\end{aligned}$$

where we have denoted $S^{\sigma,\sigma'}(\theta) := \Pi_S^{\sigma'}(\theta)S^\sigma(\theta)$. Note that $v_1 \in E_0^\sigma(\theta)$, $v_2 \in E_0^{\sigma'}(\theta)$ and $v_3 \in E_0^{\sigma''}(\theta)$ and that $\Phi^{E_0^\sigma \oplus E_0^{\sigma'}, E_0^{\sigma''}}_{\mathcal{V}_\rho^{\sigma+\sigma'}}(E_S^\sigma \oplus E_S^{\sigma'})$ is the unique vector bundle homomorphism in $\mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^{\sigma\oplus\sigma''},\ E_0^{\sigma'}\right)}$ such that $(v_1,\ v_2)$ is mapped to v_3. Hence, we begin by solving for $v,\ w$ in terms of $v_1,\ v_2$:

$$\begin{pmatrix} \mathrm{Id}_{E_0^\sigma} & S^{\sigma',\sigma} \\ S^{\sigma,\sigma'} & \mathrm{Id}_{E_0^{\sigma'}} \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix}$$

where we denoted by $\mathrm{Id}_{E_0^\sigma}$, $\mathrm{Id}_{E_0^{\sigma'}}$ the identity maps in E_0^σ, $E_0^{\sigma'}$, respectively. Hence, we obtain:

$$\Phi^{E_0^\sigma \oplus E_0^{\sigma'}, E_0^{\sigma''}}_{\mathcal{V}_\rho^{\sigma+\sigma'}}(E_S^\sigma \oplus E_S^{\sigma'}) = S^{\sigma,\sigma''}\Pi_0^\sigma \begin{pmatrix} \mathrm{Id}_{E_0^\sigma} & S^{\sigma',\sigma} \\ S^{\sigma,\sigma'} & \mathrm{Id}_{E_0^{\sigma'}} \end{pmatrix}^{-1} + S^{\sigma',\sigma''}\Pi_0^{\sigma'} \begin{pmatrix} \mathrm{Id}_{E_0^\sigma} & S^{\sigma',\sigma} \\ S^{\sigma,\sigma'} & \mathrm{Id}_{E_0^{\sigma'}} \end{pmatrix}^{-1}$$

3.1.11 Matrices for the projections

We will need to use the matrix of Π_S^σ with respect to the splitting E_0. In this subsection, we use the coordinates

$$\mathbb{C}^n(\theta) = E_0^\sigma(\theta) \oplus E_0^{\sigma' \oplus \sigma''}(\theta)$$

Note that

$$\Pi_S^{\sigma}(\theta)\begin{pmatrix}\mathrm{Id}_{E_0^{\sigma}}(\theta)\\ \\ 0\end{pmatrix}=\xi_{1,1}(\theta)\begin{pmatrix}\mathrm{Id}_{E_0^{\sigma}}(\theta)\\ \\ S^{\sigma}(\theta)\end{pmatrix},$$

$$\Pi_S^{\sigma'\oplus\sigma''}(\theta)\begin{pmatrix}\mathrm{Id}_{E_0^{\sigma}}(\theta)\\ \\ 0\end{pmatrix}=\xi_{1,2}(\theta)\begin{pmatrix}\Phi_{\mathcal{V}_\rho^{\sigma'+\sigma''}}^{E_0^{\sigma'}\oplus E_0^{\sigma''},E_0^{\sigma}}(E_S^{\sigma'}\oplus E_S^{\sigma''})(\theta)\\ \\ \mathrm{Id}_{E_0^{\sigma'\oplus\sigma''}}(\theta)\end{pmatrix},$$

$$\Pi_S^{\sigma}(\theta)\begin{pmatrix}0\\ \\ \mathrm{Id}_{E_0^{\sigma'\oplus\sigma''}}(\theta)\end{pmatrix}=\xi_{2,1}(\theta)\begin{pmatrix}\mathrm{Id}_{E_0^{\sigma}}(\theta)\\ \\ S^{\sigma}(\theta)\end{pmatrix},$$

$$\Pi_S^{\sigma'\oplus\sigma''}(\theta)\begin{pmatrix}0\\ \\ \mathrm{Id}_{E_0^{\sigma'\oplus\sigma''}}(\theta)\end{pmatrix}=\xi_{2,2}(\theta)\begin{pmatrix}\Phi_{\mathcal{V}_\rho^{\sigma'+\sigma''}}^{E_0^{\sigma'}\oplus E_0^{\sigma''},E_0^{\sigma}}(E_S^{\sigma'}\oplus E_S^{\sigma''})(\theta)\\ \\ \mathrm{Id}_{E_0^{\sigma'\oplus\sigma''}}(\theta)\end{pmatrix}$$

where $(\xi_{1,1},\ \xi_{1,2},\ \xi_{2,1},\ \xi_{2,2})$ is the solution of the linear system

$$\xi_{1,1}(\theta)\begin{pmatrix}\mathrm{Id}_{E_0^{\sigma}}(\theta)\\ \\ S^{\sigma}(\theta)\end{pmatrix}+\xi_{1,2}(\theta)\begin{pmatrix}\Phi_{\mathcal{V}_\rho^{\sigma'+\sigma''}}^{E_0^{\sigma'}\oplus E_0^{\sigma''},E_0^{\sigma}}(E_S^{\sigma'}\oplus E_S^{\sigma''})(\theta)\\ \\ \mathrm{Id}_{E_0^{\sigma'\oplus\sigma''}}(\theta)\end{pmatrix}=\begin{pmatrix}\mathrm{Id}_{E_0^{\sigma}}(\theta)\\ \\ 0\end{pmatrix}$$

$$\xi_{2,1}(\theta)\begin{pmatrix}\mathrm{Id}_{E_0^{\sigma}}(\theta)\\ \\ S^{\sigma}(\theta)\end{pmatrix}+\xi_{2,2}(\theta)\begin{pmatrix}\Phi_{\mathcal{V}_\rho^{\sigma'+\sigma''}}^{E_0^{\sigma'}\oplus E_0^{\sigma''},E_0^{\sigma}}(E_S^{\sigma'}\oplus E_S^{\sigma''})(\theta)\\ \\ \mathrm{Id}_{E_0^{\sigma'\oplus\sigma''}}(\theta)\end{pmatrix}=\begin{pmatrix}0\\ \\ \mathrm{Id}_{E_0^{\sigma'\oplus\sigma''}}(\theta)\end{pmatrix}$$

Hence,

$$\begin{pmatrix}\xi_{1,1} & \xi_{1,2}\\ \\ \xi_{2,1} & \xi_{2,2}\end{pmatrix}=\begin{pmatrix}\mathrm{Id}_{E_0^{\sigma}} & \Phi_{\mathcal{V}_\rho^{\sigma'+\sigma''}}^{E_0^{\sigma'}\oplus E_0^{\sigma''},E_0^{\sigma}}(E_S^{\sigma'}\oplus E_S^{\sigma''})\\ \\ S^{\sigma} & \mathrm{Id}_{E_0^{\sigma'\oplus\sigma''}}\end{pmatrix}^{-1}$$

3.1.12 Cocycles and Adapted Cocycles over T_ω

Given $A \in \mathcal{A}_{\rho,\ \omega}^{GL(\mathbb{C}^n \times \mathbb{T}^d_\rho)}$ and $k \in \mathbb{Z}$, then $A^{(k)} \in \mathcal{A}_{\rho,\ k\omega}^{GL(\mathbb{C}^n \times \mathbb{T}^d_\rho)}$ is defined by equation (10).

Let E_S be a splitting, and $\sigma, \sigma' \in \{s, u, c\}$. We will use the *adapted* to E_S cocycles $A^{\sigma,\sigma'}[S] \in \mathcal{A}_{\rho,\ \omega}^{\mathcal{L}\left(\mathbb{C}^n \times \mathbb{T}^d_\rho\right)}$, defined as follows:

$$A^{\sigma,\sigma'}[S] \ := \ \Pi_S^{\sigma'} \circ T_\omega \ A \ \Pi_S^\sigma$$

We endow $A^{\sigma,\sigma}[S]$ with the structure of a cocycle as follows: Given $k \in \mathbb{N}$, let

$$\begin{aligned}
(A[S]^{\sigma,\sigma})^{(k)} \ &:= \ \Pi_S^\sigma \circ T_{k\omega} \ \ A \circ T_{(k-1)\omega} \ \ \Pi_S^\sigma \circ T_{(k-1)\omega} \ \ \ldots \\
&\ldots \ \ \Pi_S^\sigma \circ T_\omega \ A \ \Pi_S^\sigma \\
(A[S]^{\sigma,\sigma})^{(-k)} \ &:= \ \Pi_S^\sigma \circ T_{-k\omega} \ \ A^{-1} \circ T_{-(k-1)\omega} \ \ \Pi_S^\sigma \circ T_{-(k-1)\omega} \ \ \ldots \\
&\ldots \ \ \Pi_S^\sigma \circ T_{-\omega} \ A^{-1} \ \Pi_S^\sigma
\end{aligned}$$

Remark 8. *Note that $A \in \mathcal{A}_{\rho,\ \omega}^{GL(\mathbb{C}^n \times \mathbb{T}^d_\rho)}$ acts on $\mathcal{A}_\rho^{G_n^\sigma}$ by left-multiplication: Given $E_S^\sigma \in \mathcal{A}_\rho^{G_n^\sigma}$, we define $A\ E_S^\sigma \in \mathcal{A}_\rho^{G_n^\sigma}$ by*

$$A\ E_S^\sigma(\theta) := \ \{\ Av \mid v \in E_S^\sigma(\theta)\ \}\ , \quad \theta \in \mathbb{T}^d_\rho$$

For $\sigma \neq \sigma'$ it holds that

$$\left|\ A^{\sigma,\sigma'}[S]\ \right|_\rho \leq C \mid A \mid_\rho \ d_{\mathcal{V}_\rho^\sigma}\left(E_S^\sigma \circ T_\omega,\ A\ E_S^\sigma\right)$$

3.1.13 Whiskered splittings

Given $\mathtt{N} \in \mathbb{N}$, to remain fixed throughout the iterative procedure outlined in Algorithm 1, and positive numbers $\mu_s,\ \mu_u,\ \mu_{c,+},\ \mu_{c,-}$ satisfying

$$\mu_s < 1,\ \mu_u < 1, \ \ 1 < \mu_{c,+},\ 1 < \mu_{c,-}, \ \ \mu_s\mu_{c,+} < 1,\ \mu_u\mu_{c,-} < 1,$$

we will say that the splitting E_S is $\mathtt{N}, \mu_s, \mu_u, \mu_{c,+}, \mu_{c,-}$- *whiskered with respect to* the cocycle A if there holds:

$$\left| \, (A^{s,s}[S])^{(\mathtt{N})} \, \right|_\rho \leq \mu_s^{\mathtt{N}}, \qquad \left| \, (A^{u,u}[S])^{(-\mathtt{N})} \, \right|_\rho \leq \mu_u^{\mathtt{N}},$$

$$\begin{cases} \left| \, (A^{c,c}[S])^{(\mathtt{N})} \, \right|_\rho \leq \mu_{c,+}^{\mathtt{N}}, \\ \\ \left| \, (A^{c,c}[S])^{(-\mathtt{N})} \, \right|_\rho \leq \mu_{c,-}^{\mathtt{N}} \end{cases} \tag{20}$$

Remark 9. *Let*

$$C_h \; := \; C_h(\mathtt{N}, \mu_s, \; \mu_u, \; \mu_{c,+}, \; \mu_{c,-}, \; A, \; E_S) := \max_{\substack{\sigma \, \in \, \{s,u,c\}, \\ -\mathtt{N} \, \leq \, j \, \leq \, \mathtt{N}}} \left| \, (A^{\sigma,\sigma}[S])^{(j)} \, \right|_\rho \tag{21}$$

Then, for $k\mathtt{N} \; \leq \; n \; < (k+1)\mathtt{N}$*, we have*

$$\begin{aligned} \left| \, (A^{\sigma,\sigma}[S])^{(n)} \, \right|_\rho &\leq \left| \, (A^{\sigma,\sigma}[S])^{(k\mathtt{N})} \, \right|_\rho \left| \, (A^{\sigma,\sigma}[S])^{(n-k\mathtt{N})} \, \right|_\rho \\ &\leq \mu_\sigma^{k\mathtt{N}} \, \widetilde{C}_h \\ &\leq C_h \, \mu_\sigma^n \end{aligned}$$

Hence, if E_S *is* $\mathtt{N}, \mu_s, \mu_u, \mu_{c,+}, \mu_{c,-}$- whiskered with respect to *the cocycle* A*, it will also hold that, given* $n \in \mathbb{N}$*,*

$$\left| \, (A^{s,s}[S])^{(n)} \, \right|_\rho \leq C_h \, \mu_s^n, \qquad \left| \, (A^{u,u}[S])^{(-n)} \, \right|_\rho \leq C_h \, \mu_u^n,$$

$$\begin{cases} \left| \, (A^{c,c}[S])^{(n)} \, \right|_\rho \leq C_h \, \mu_{c,+}^n, \\ \\ \left| \, (A^{c,c}[S])^{(-n)} \, \right|_\rho \leq C_h \, \mu_{c,-}^n \end{cases} \tag{22}$$

Conversely, given any $\epsilon > 0$, if inequalities (22) *hold then inequalities* (20) *hold for any* $\mathtt{N}_\epsilon \geq 1/\epsilon$, *and the slightly deteriorated exponent* $\widetilde{\mu}_\sigma := \mu_\sigma + \epsilon$.

We will use the inequalities (22) *rather than inequalities* (20) *in the definition of whiskered splitting. To make reference to the constant C_h we will say that the splitting E_S is* $\mathtt{N}, C_h, \ \mu_s, \mu_u, \mu_{c,+}, \mu_{c,-}$- whiskered with respect to *the cocycle A, where* $C_h := C_h(\mathtt{N}, \mu_s, \ \mu_u, \ \mu_{c,+}, \ \mu_{c,-}, \ A, \ E_S)$ *is computed as in* (21).

Remark 10. *The definition of whiskered splitting presented in Subsection 2.3.3 (which used directly the cocycle A instead of the adapted cocycles $A^{\sigma,\sigma}[S]$) is more common in the literature. It is appropriate for invariant splittings, but it is too strong a condition for* approximately *invariant splittings. We mention briefly why:*

Fix $\theta \in \mathbb{T}^d_\rho$ and let, e.g. $v \in E^s_S(\theta)$, with $v \neq 0$. Then, if E_S is only approximately invariant, it can occur that

$$\Pi^u_S(\theta + \omega) \ \ (A(\theta) \ v) \neq 0$$

(of course, this cannot occur if E_S is invariant under A). Then, since the base $\mathbb{T}^d_\rho$ is compact, it is straightforward to show that

$$\Big| \ A^{(n)} \ v \ \Big|_\rho \ \geq \ C \ C_h \ \mu^n_u \ |v| \ , \quad n > 0$$

where the geometric constant $C > 0$ depends on the minimum angle between the bundles E^s_S and E^u_S over $\mathbb{T}^d_\rho$. Hence the condition, stated in Subsection 2.3.3,

$$\Big| \ A^{(n)} \ v \ \Big|_\rho \ \leq C_h \ \mu^n_s \ |v|, \quad v \in E^s_S$$

in general will hold only if $v = 0$.

The condition in equation 22 can be thought as the appropriate relaxation for the case of approximately invariant splittings of the definition of whiskered invariant *splittings in Subsection 2.3.3. We mention that no essential changes will be needed in the arguments concerning the partial hyperbolicity of the splitting E_S, with respect to the (by now standard) techniques found in the literature.*

3.1.14 Number Theoretic conditions.

Let $\nu,\ \tau > 0$ to remain fixed throughout the iterative procedure. In solving the cohomology equations that appear at each step of Algorithm 1, we will need to assume that the tangential frequency $\omega \in \mathbb{T}^d$ satisfies the following non-resonance condition:

Diophantine condition: $\omega \in DC(\nu, \tau)$ if

$$\mid j - k \cdot \omega \mid \ \geq \ \nu \mid k \mid^{-\tau}, \quad k \in \mathbb{Z}^d \setminus \{\vec{0}\},\ j \in \mathbb{Z}$$

We will also need to assume that the normal frequency $\Lambda = (\lambda_1,\ \lambda_2, \ldots, \lambda_c) \in \mathbb{C}^c$ and ω satisfy the following joint non-resonance conditions:

First Melnikov condition: $\Lambda \in DC_\omega^{1^{\rm st}}(\nu, \tau)$ if for $1 \leq j \leq c$:

- $\mid \lambda_j - \exp(2\pi i k \cdot \omega) \mid \ \geq \ \nu \mid k \mid^{-\tau}, \quad k \in \mathbb{Z}^d \setminus \{\vec{0}\}$

- $\lambda_j \neq 1$

Second Melnikov condition: $\Lambda \in DC_\omega^{2^{\rm nd}}(\nu, \tau)$ if for $1 \leq\ l,\ j \leq c$:

- $\mid \lambda_l - \lambda_j \exp(2\pi i k \cdot \omega) \mid \geq \nu \mid k \mid^{-\tau}, \quad k \in \mathbb{Z}^d \setminus \{\vec{0}\}$

- $\lambda_j \neq \lambda_l$ if $j \neq l$

CHAPTER IV

STATEMENT OF THE MAIN RESULTS AND OUTLINE OF THEIR PROOFS

In this Chapter we state our main result, Theorem 1. Theorem 1 is formulated in an *a posteriori* format: The objects to be computed are zeroes of suitable functional equations, which we introduce in Subsection 4.1. An outline of the proof of Theorem 1 is presented in Section 4.3. The full proof of Theorem 1 takes up Chapters 5, 6 and 7.

4.1 Invariance and Reducibility equations.

We now introduce five functional equations, involving the functionals $\mathscr{I}$ (see Subsection 4.1.1), $\mathscr{S}^\sigma$ (see Subsection 4.1.2), and $\mathscr{R}$ (see Subsection 4.1.3). The motivation for these functional equations will be discussed in Subsection 4.1.4.

4.1.1 Invariance equation for K.

We start with the functional associated to the invariance of the torus K, which is:

$$\mathscr{I} : \mathcal{A}_\rho^{\mathbb{C}^n} \times \mathbb{C}^c \longrightarrow \mathcal{A}_\rho^{\mathbb{C}^n}$$

$$\mathscr{I}[K,\beta] \;=\; f_\beta \circ K \;-\; K \circ T_\omega \;,$$

If $\mathscr{I}[K,\beta] \equiv 0$, then the graph of K over $\mathbb{T}^d_\rho$, $\mathfrak{T}_K$ (see (3)), is invariant under F_β, where:

$$\begin{aligned} F_\beta \,:\; \mathbb{T}^d_\rho \times \mathbb{C}^n &\to \mathbb{T}^d_\rho \times \mathbb{C}^n \\ (\theta, z) &\mapsto (\theta+\omega, f_\beta(\theta,z)). \end{aligned}$$

If $\;|\,\mathscr{I}[K,\beta]\,|_\rho \ll 1$, then we say that $\mathfrak{T}_K$ is approximately invariant under F_β.

Remark 11. *For the functional $\mathscr{I}$ to be well-defined at $(K,\ \beta)$, we need to impose that the image of $\mathbb{T}^d_\rho$ under K (which we will denote by $K(\mathbb{T}^d_\rho)$) is contained in the domain of analyticity of f_β.*

We will need to ensure, at each step of the iterative procedure outlined in Algorithm 1, that this condition is verified. Informally, since the corrections produced at each step of Algorithm 1 can be made small with the error of the initial approximate solutions (see the discussion on tame estimates in Section 4.3), the condition $K(\mathbb{T}^d_\rho) \subset Domain(f_\beta)$ will be automatically satisfied throughout the iterative procedure if the initial errors are sufficiently small.

4.1.2 Invariance equation for the splitting.

Let $\sigma, v \in \{s, c, u\}$, $\{\sigma,\ \sigma'\ , \sigma''\}$ a permutation of $\{s,\ u,\ c\}$, $S^\sigma \in \mathcal{A}_{\rho,\ 0}^{\mathcal{L}(\ E_0^{s\oplus u\oplus c},\ E_0^{s\oplus u\oplus c}\ )}$ and let

$$E^\sigma_S := \left(\Phi_{\mathcal{V}^\sigma_\rho}^{E^\sigma_0, E_0^{\sigma'\oplus\sigma''}}\right)^{-1}(S^\sigma)$$

be the analytic vector bundle that corresponds to $S^\sigma := \left(\ S^{\sigma,\sigma'}, S^{\sigma,\sigma''}\ \right)$ under the parameterization of $\mathcal{V}^\sigma_\rho$ introduced in Subsection 3.1.8. We had introduced the adapted to E_S cocycles $A^{\sigma,v}_{K,\beta}\ [S]$ in Subsection 3.1.12. We will use the shorthand notation

$$A^{\sigma,v}_{K,\beta}\ :=\ A^{\sigma,v}_{K,\beta}\ [0]\ =\ \Pi^v_0 \circ T_\omega\ \ A_{K,\beta}\ \ \Pi^\sigma_0$$

Recall from Subsection 3.1.8 that E^σ_S is modelled after E^σ_0 via the parameterization

$$v \mapsto v + S^{\sigma,\sigma'}v + S^{\sigma,\sigma''}v, \quad v \in E^\sigma_0$$

Here, $S^{\sigma,\sigma'} = \Pi^{\sigma'}_0 S^\sigma$.

Thus, E^σ_S is $A_{K,\beta}$-invariant if, and only if, given any $v \in \mathcal{A}^{E^\sigma_0}_{\rho,0}$, there exists $w_v \in \mathcal{A}^{E^\sigma_0}_{\rho,\ \omega}$ such that:

$$\begin{pmatrix} A_{K,\beta}^{\sigma,\sigma} & A_{K,\beta}^{\sigma',\sigma} & A_{K,\beta}^{\sigma'',\sigma} \\ \\ A_{K,\beta}^{\sigma,\sigma'} & A_{K,\beta}^{\sigma',\sigma'} & A_{K,\beta}^{\sigma'',\sigma'} \\ \\ A_{K,\beta}^{\sigma,\sigma''} & A_{K,\beta}^{\sigma',\sigma''} & A_{K,\beta}^{\sigma'',\sigma''} \end{pmatrix} \begin{pmatrix} v \\ \\ S^{\sigma,\sigma'} v \\ \\ S^{\sigma,\sigma''} v \end{pmatrix} = \begin{pmatrix} w_v \\ \\ S^{\sigma,\sigma'} \circ T_\omega \; w_v \\ \\ S^{\sigma,\sigma''} \circ T_\omega \; w_v \end{pmatrix} \tag{23}$$

System (23) holds if, and only if,

$$A_{K,\beta}^{\sigma,\sigma'} + A_{K,\beta}^{\sigma',\sigma'} \; S^{\sigma,\sigma'} + A_{K,\beta}^{\sigma'',\sigma'} \; S^{\sigma,\sigma''} = S^{\sigma,\sigma'} \circ T_\omega \left[A_{K,\beta}^{\sigma,\sigma} + A_{K,\beta}^{\sigma',\sigma} \; S^{\sigma,\sigma'} + A_{K,\beta}^{\sigma'',\sigma} S^{\sigma,\sigma''} \right]$$

$$A_{K,\beta}^{\sigma,\sigma''} + A_{K,\beta}^{\sigma',\sigma''} \; S^{\sigma,\sigma'} + A_{K,\beta}^{\sigma'',\sigma''} \; S^{\sigma,\sigma''} = S^{\sigma,\sigma''} \circ T_\omega \left[A_{K,\beta}^{\sigma,\sigma} + A_{K,\beta}^{\sigma',\sigma} \; S^{\sigma,\sigma'} + A_{K,\beta}^{\sigma'',\sigma} S^{\sigma,\sigma''} \right]$$

Thus, the functional associated to the invariance of the bundle E_S^σ is:

$$\mathscr{S}^\sigma : \mathcal{A}_\rho^{\mathbb{C}^n} \times \mathbb{C}^c \times \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma, E_0^{\sigma'}\right)} \times \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_0^\sigma, E_0^{\sigma''}\right)} \longrightarrow \mathcal{A}_{\rho,\ \omega}^{\mathcal{L}\left(E_0^\sigma, E_0^{\sigma'}\right)} \times \mathcal{A}_{\rho,\ \omega}^{\mathcal{L}\left(E_0^\sigma, E_0^{\sigma''}\right)}$$

$$\mathscr{S}^\sigma \left[K, \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''} \right] =$$

$$\begin{pmatrix} A_{K,\beta}^{\sigma,\sigma'} + A_{K,\beta}^{\sigma',\sigma'} \; S^{\sigma,\sigma'} + A_{K,\beta}^{\sigma'',\sigma'} \; S^{\sigma,\sigma''} - S^{\sigma,\sigma'} \circ T_\omega \left[A_{K,\beta}^{\sigma,\sigma} + A_{K,\beta}^{\sigma',\sigma} \; S^{\sigma,\sigma'} + A_{K,\beta}^{\sigma'',\sigma} S^{\sigma,\sigma''} \right] \\ \\ A_{K,\beta}^{\sigma,\sigma''} + A_{K,\beta}^{\sigma',\sigma''} \; S^{\sigma,\sigma'} + A_{K,\beta}^{\sigma'',\sigma''} \; S^{\sigma,\sigma''} - S^{\sigma,\sigma''} \circ T_\omega \left[A_{K,\beta}^{\sigma,\sigma} + A_{K,\beta}^{\sigma',\sigma} \; S^{\sigma,\sigma'} + A_{K,\beta}^{\sigma'',\sigma} S^{\sigma,\sigma''} \right] \end{pmatrix}$$

If $\mathscr{S}^\sigma[K, \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''}] \equiv 0$, then E_S^σ is invariant under $A_{K,\beta}$. If

$$\Big| \; \mathscr{S}^\sigma[K, \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''}] \; \Big|_\rho \ll 1$$

we will say that E_S^σ is approximately invariant under $A_{K,\ \beta}$.

Remark 12. *For the functionals $\mathscr{S}^\sigma$ to be well-defined at $\left(K, \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''}\right)$, the projected cocycles $A_{K,\beta}^{\sigma,\sigma'}$ must be well-defined. This, in turn, requires that E_S is a splitting (so that the projections onto the bundles E_S^σ are well-defined). Hence, at each* *step of the iterative procedure outlined in Algorithm 1, it must be verified that E_S^σ is a splitting, which is an open condition. As in Remark 11, it will hold automatically throughout the iterative procedure if the initial errors are sufficiently small.*

We now show that the approximate invariance of the bundle E_S^σ under $A_{K,\beta}$ just defined admits a geometric interpretation:

Proposition 1. *Denote $\mathscr{S}^\sigma[K, \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''}] \ = \ e^\sigma_{\mathscr{S}}$ and let*

$$E_S^\sigma := \left(\Phi_{\mathcal{V}_\rho^\sigma}^{E_0^\sigma, E_0^{\sigma'\oplus\sigma''}} \right)^{-1} (S^\sigma)$$

be the analytic vector bundle that corresponds to $S^\sigma := \left(\ S^{\sigma,\sigma'}, S^{\sigma,\sigma''} \ \right)$ under the parameterization of $\mathcal{V}_\rho^\sigma$ introduced in Subsection 3.1.8. Then:

1. *$d_{\mathcal{V}_\rho^\sigma}(E_S^\sigma \circ T_\omega, \ A_{K,\beta} \ E_S^\sigma) < C \ \mid e^\sigma_{\mathscr{S}} \mid_\rho$ (we introduced the vector bundle $A_{K,\beta} \ E_S^\sigma$ in Remark 8).*

2. *$\left| \ A_{K,\beta}^{\sigma,\sigma'}[S] \ \right|_\rho < C \ \left|A_{K,\ \beta}\right|_\rho \mid e^\sigma_{\mathscr{S}} \mid_\rho$, for $\sigma \neq \sigma'$.*

Proof. Let $e^\sigma_{\mathscr{S}} = \left(\ e^{\sigma,\sigma'}_{\mathscr{S}}, \ e^{\sigma,\sigma''}_{\mathscr{S}} \ \right)^t$. Given any $\theta \in \mathbb{T}^d_\rho$ and any $\bar{v} \in E_S^\sigma(\theta)$, normalizing if necessary, let $v \in E_S^\sigma(\theta)$ be such that $|v_A| = 1$, where $v_A := v_A(v) \in A_{K,\beta}(\theta) \ E_S^\sigma(\theta)$ is defined as

$$v_A \ := \ A_{K,\beta}(\theta) \ \left(\ v + S^{\sigma,\sigma'}(\theta) \ v + S^{\sigma,\sigma''}(\theta) \ v \ \right)^t$$

Note that $|v| \ \leq \ C \left| \ A_{K,\beta}^{-1}(\theta) \ \right|$. Since $A_{K,\beta}(\theta)$ is an isomorphism, given any $x \in A_{K,\beta}(\theta) \ E_S^\sigma(\theta)$ with $|x| = 1$, there exists $v \in E_S^\sigma(\theta)$ such that $v_A(v) = x$. Let $w_v \in E_0^\sigma(\theta + \omega)$ be defined as

$$w_v := A_{K,\beta}^{\sigma,\sigma}(\theta) \ v + A_{K,\beta}^{\sigma',\sigma}(\theta) \ S^{\sigma,\sigma'}(\theta) \ v + A_{K,\beta}^{\sigma'',\sigma}(\theta) \ S^{\sigma,\sigma''}(\theta) \ v$$

Note that $\mathscr{S}^\sigma[K, \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''}] \ = \ e^\sigma_{\mathscr{S}}$ implies

$$v_A = \widetilde{w_v} + v_e$$

where

$$\widetilde{w_v} := \begin{pmatrix} w_v \\ \\ S^{\sigma,\sigma'}(\theta+\omega)\ w_v \\ \\ S^{\sigma,\sigma''}(\theta+\omega)\ w_v \end{pmatrix}, \qquad v_e := \begin{pmatrix} 0 \\ \\ e^{\sigma,\sigma'}_{\mathscr{S}}(\theta)\ v \\ \\ e^{\sigma,\sigma'}_{\mathscr{S}}(\theta)\ v \end{pmatrix}$$

we remark that $\widetilde{w_v} \in E^{\sigma}_S(\theta+\omega)$. Since $v_A \in A_{K,\beta}(\theta)\ E^{\sigma}_S(\theta)$ and v is arbitrary, we have

$$\begin{aligned} d_{G^{\sigma}_n}\left(A_{K,\beta}(\theta)\ E^{\sigma}_S(\theta),\ E_S(\theta+\omega)\right) &\leq |v_A - \widetilde{w_v}| \\ &\leq |v_e| \\ &\leq C\left|A^{-1}_{K,\beta}(\theta)\right|\ |e^{\sigma}_{\mathscr{S}}(\theta)| \end{aligned}$$

Since θ is arbitrary, Item (1) follows. Item (2) follows from item (1) and Remark 8. □

Remark 13. *Note that for the splitting E_0 we have:*

$$\mathscr{S}^{\sigma}[K_0,\beta_0,0,0] = \begin{pmatrix} A^{\sigma,\sigma'}_{K_0,\beta_0} \\ \\ A^{\sigma,\sigma''}_{K_0,\beta_0} \end{pmatrix}$$

Thus, in the statement of Theorem 1, we will write

$$\left|\ A^{\sigma,\sigma'}_{K_0,\beta_0}\ \right|_{\rho_0} < \epsilon, \qquad \left|\ A^{\sigma,\sigma''}_{K_0,\beta_0}\ \right|_{\rho_0} < \epsilon$$

instead of $|\ \mathscr{S}^s\,[K_0,\beta_0,0,0]\ |_{\rho_0} < \epsilon$.

4.1.3 Reducibility equation.

Given a whiskered splitting E_S, we denote the restriction of Π^c_0 to E^c_S by $\Pi^c_0|_{E^c_S} \in \mathcal{A}^{\mathcal{L}\left(E^c_S,\ E^c_0\ \right)}_{\rho,\ 0}$. Note that $\Pi^c_0|_{E^c_S}$ is a vector bundle isomorphism, and thus admits an inverse

$$\left(\ \Pi^c_0|_{E^c_S}\ \right)^{-1} \in \mathcal{A}^{\mathcal{L}\left(E^c_0,\ E^c_S\ \right)}_{\rho,\ 0}$$

We will assume that the center bundle E_0^c is trivializable via an analytic vector bundle isomorphism $\psi_0 \in \mathcal{A}_{\rho,\,0}^{\mathcal{L}\left(\mathbb{C}^c \times \mathbb{T}^d_\rho,\; E_0^c\;\right)}$. We can thus define an analytic vector bundle isomorphism $\psi_S \in \mathcal{A}_{\rho,\,0}^{\mathcal{L}\left(\mathbb{C}^c \times \mathbb{T}^d_\rho,\; E_S^c\;\right)}$ by

$$\psi_S := \left(\; \Pi_0^c|_{E_S^c} \;\right)^{-1}\; \psi_0$$

Given $S^\sigma \in \mathcal{A}_{\rho,\,0}^{\mathcal{L}\left(\;E_0^\sigma,\; E_0^{\sigma'\oplus\sigma''}\right)}$, where $\{\sigma,\; \sigma',\; \sigma''\}$ is a permutation of $\{s, u, c\}$, let $S \in \prod_{\sigma\in\{s,u,c\}} \mathcal{A}_{\rho,\,0}^{\mathcal{L}\left(\;E_0^\sigma,\; E_0^{\sigma'\oplus\sigma''}\right)}$ be defined as

$$S := (S^s,\; S^u,\; S^c\;)$$

We can now define $\overline{A_{K,\beta}^{c,c}}[S] \in \mathcal{A}_\rho^{GL(\mathbb{C}^c)}$, the reduced derivative cocycle along the central directions, as

$$\overline{A_{K,\beta}^{c,c}}[S] := \psi_S^{-1} \circ T_\omega \;\; A_{K,\beta}^{c,c}[S] \;\; \psi_S$$

The functional associated to the reducibility of $\overline{A_{K,\beta}^{c,c}}[S]$ along the central directions is:

$$\mathscr{R} \;:\; \mathcal{A}_\rho^{\mathbb{C}^n} \times \mathbb{C}^c \times \prod_{\sigma\in\{s,u,c\}} \mathcal{A}_{\rho,\,0}^{\mathcal{L}\left(\;E_0^\sigma,\; E_0^{\sigma'\oplus\sigma''}\right)} \times \mathcal{A}_\rho^{GL(\mathbb{C}^c)} \longrightarrow \mathcal{A}_\rho^{\mathbb{C}^{c\times c}}$$

$$\mathscr{R}[K,\beta,S,U] := \overline{A_{K,\beta}^{c,c}}[S]\; U - U \circ T_\omega\; \Lambda$$

If $\mathscr{R}[K,\beta,S,U] \equiv 0$, then the cocycle $A_{K,\beta}$ is reducible to Λ via U. If $|\; \mathscr{R}[K,\beta,S,U]\;|_\rho \ll 1$, then we will say that the cocycle $A_{K,\beta}$ is approximately reducible to Λ via U.

4.1.4 Motivation for the definition of the functionals $\mathscr{I}$, $\mathscr{S}^\sigma$, $\mathscr{R}$.

The main goal of our study is the invariance equation for K:

$$\mathscr{I}[K,\;\beta] \;\equiv\; 0$$

which gives the existence of an invariant torus. As we will see, when we linearize the functional $\mathscr{I}$ we are lead to a non-local, non-constant coefficient equation. The functional equations associated to the invariance of the whiskered splitting S are

$$\mathscr{S}^\sigma[K,\beta,S^{\sigma,\sigma'},S^{\sigma,\sigma''}] \;\equiv\; 0$$

The functional equation associated to the reducibility of the projected cocycle $A^{c,c}_{K,\beta}$ is

$$\mathscr{R}[K,\beta,S,U] \equiv 0$$

One can think of the functional equations associated to $\mathscr{S}$ and $\mathscr{R}$ as auxiliary equations that allow to solve the linearized equation for $\mathscr{I}$. As pointed out in Proposition 1, the functional equations associated to the invariance of the whiskered splitting S also has a geometric meaning and solving it gives geometric information about the dynamics near the torus.

4.2 Statement of Theorem 1

Assume given:

- ρ_0, η_0, ν, τ positive constants, and $\mathtt{N} \in \mathbb{N}$.

- $C_{h,0},\ \mu_{s,0},\ \mu_{u,0},\ \mu_{c,+,0},\ \mu_{c,-,0}$ positive constants such that:

$$\mu_{s,0} < 1,\ \mu_{u,0} < 1,\ 1 < \mu_{c,+,0},\ 1 < \mu_{c,-,0},\ \mu_{s,0}\ \mu_{c,+,0} < 1,\ \mu_{u,0}\ \mu_{c,-,0} < 1$$

- $K_0 \in \mathcal{A}_{\rho_0}^{\mathbb{C}^n}$, and a parameter $\beta_0 \in \mathbb{C}^c$.

- $f \in \mathcal{A}^{\mathbb{C}^n}_{\mathcal{D}_{\rho_0,3\eta_0}(K_0,\ \beta_0)}$. We will use the notation $f_\beta(\theta, z) := f(\theta, z, \beta)$. We denote, as before,

$$A_{K_0,\beta_0} \ := \ D_2 f_{\beta_0} \circ K_0$$

  We will assume that $A_{K_0,\beta_0} \in \mathcal{A}_\rho^{GL(\mathbb{C}^c)}$.

- $E_0 = (E_0^s,\ E_0^u,\ E_0^c) \in \mathcal{A}_{\rho_0}^{G_n^s \times G_n^u \times G_n^c}$ a $\mathtt{N},\ C_{h,0},\ \mu_{s,0}, \mu_{u,0}, \mu_{c,+,0},\ \ \mu_{c,-,0}$-whiskered analytic splitting (with respect to A_{K_0,β_0}) over $\mathbb{T}^d_{\rho_0}$.

- $\eta_0^{\mathscr{S}}$ a positive constant, and $\mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^\sigma)$, $\sigma \in \{s, u, c\}$, a neighborhood of E_0^σ in $\mathcal{U}^{\eta_0^{\mathscr{S}}}_{\mathcal{V}_\rho^\sigma}(E_0^\sigma)$ (see Subsection 3.1.8) such that
  - If $E^\sigma \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^\sigma)$, $\sigma \in \{s, u, c\}$, then $(E^s,\ E^u,\ E^c\ )$ is also a splitting.
  - If $d_{\mathcal{U}^{\eta_0^{\mathscr{S}}}_{\mathcal{V}_\rho^\sigma}(E_0^\sigma)}(E_0^\sigma,\ E^\sigma\ )\ <\ \eta_0^{\mathscr{S}}$, then $E^\sigma \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^\sigma)$.

- An analytic vector bundle isomorphism $\psi_0 \in \mathcal{A}_{\rho_0,\ 0}^{\mathcal{L}\left(\mathbb{C}^c \times \mathbb{T}^d_{\rho_0},\ E_0^c\ \right)}$

- $U_0 \in \mathcal{A}_{\rho_0}^{\mathbb{C}^{c\times c}}$, $\Lambda = \mathrm{diag}[\lambda_1,\ \lambda_2,\ \dots, \lambda_c] \in GL(c, \mathbb{C})$, $\omega \in \mathbb{T}^d$ such that

$$\omega \in DC(\nu, \tau),$$

$$\mathrm{Spec}(\Lambda) \in DC_\omega^{1st}(\nu, \tau) \cap DC_\omega^{2nd}(\nu, \tau)$$

Theorem 1. *Let $\eta^{\daleth}_{\mathscr{R},0}$ be the affine function defined in Lemma 9, and let $\pi_{Diag} : \mathbb{C}^{c\times c} \to \mathbb{C}^c$ be the map that returns the diagonal entries of a matrix:*

$$\pi_{Diag}(B)_j = B_{jj}\ ,\quad B \in \mathbb{C}^{c\times c},\quad 1 \le j \le c$$

and let $\pi_{Diag} \circ \left[\mathtt{Lin}(U_0^{(-1)} \circ T_\omega\ \eta^{\daleth}_{\mathscr{R},0})\right]^{\frown}(0)\ : \mathbb{C}^c \to \mathbb{C}^c$ be the linear map defined by the compositions:

$$\mathbb{C}^c \xrightarrow{\mathtt{Lin}(U_0^{(-1)} \circ T_\omega\ \eta^{\daleth}_{\mathscr{R},0})} \mathcal{A}_{\rho_0}^{\mathbb{C}^{c\times c}} \xrightarrow{\ ^\frown(0)\ } \mathbb{C}^{c\times c} \xrightarrow{\pi_{Diag}} \mathbb{C}^c$$

(we have denoted by $^\frown(0)$ the linear map $\mathcal{A}_{\rho_0}^{\mathbb{C}^{c\times c}} \to \mathbb{C}^{c\times c}$ that, for $f \in \mathcal{A}_{\rho_0}^{\mathbb{C}^{c\times c}}$, returns $f^\frown(0)$).

If the following non-degeneracy condition holds:

$$\pi_{Diag} \circ \left[\mathtt{Lin}(U_0^{(-1)} \circ T_\omega\ \eta^{\daleth}_{\mathscr{R},0})\right]^{\frown}(0)\ \in\ GL(c, \mathbb{C}) \tag{24}$$

let

$$\left|\ \left(\ \pi_{Diag} \circ \left[\mathtt{Lin}(U_0^{(-1)} \circ T_\omega\ \eta^{\daleth}_{\mathscr{R},0})\right]^{\frown}(0)\ \right)^{-1}\ \right| \ :=\ M_0 \tag{25}$$

Then, there exists

$$\epsilon_0 > 0$$

where ϵ_0 *can be computed from the initial data* $\rho_0,\ \eta_0,\ \nu,\ \tau,\ C_{h,0},\ \mu_{s,0},\ \mu_{u,0}, \mu_{c,+,0},\ \mu_{c,-,0},\ K_0,\ \beta_0,\ f,$ $\sigma \in \{s,u,c\},\ \psi_0,\ U_0,\ \omega,,\ \Lambda$ *and* M_0*, such that, if for some* $\epsilon \leq \epsilon_0$ *it holds that* [1]*:*

- $|\ \mathscr{I}\,[K_0,\beta_0]\ |_{\rho_0} < \epsilon.$

- $|\ \mathscr{R}\,[\ K_0,\ \beta_0,\ 0,\ U_0\ ]\ |_{\rho_0} < \epsilon.$

- *for each* $\{\sigma,\sigma'\} \subset \{s,u,c\}$, $\sigma \neq \sigma'$,

$$\left|\ A^{\sigma,\ \sigma'}_{K_0,\beta_0}\ \right|_{\rho_0} < \epsilon$$

Then, there exist

- $\rho_\infty,\ C,\ C_{h_\infty},\ \mu_{s,\infty},\ \mu_{u,\infty},\ \mu_{c,+,\infty},\ \mu_{c,-,\infty}$ *positive constants such that:*

  $$\mu_{s,\infty} < 1,\ \mu_{u,\infty} < 1,\ 1 < \mu_{c,+,\infty},\ 1 < \mu_{c,-,\infty},\ \mu_{s,\infty}\ \mu_{c,+,\infty} < 1,\ \mu_{u,\infty}\ \mu_{c,-,\infty} < 1$$

- $K_\infty \in \mathcal{A}^{\mathbb{C}^n}_{\rho_\infty}$, *and a parameter* $\beta_\infty \in \mathbb{C}^c$.

- $U_\infty \in \mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho_\infty}$.

- $E_{S_\infty} = \left(E^s_{S_\infty},\ E^u_{S_\infty},\ E^c_{S_\infty}\right) \in \mathcal{A}^{G^s_n \times G^u_n \times G^c_n}_{\rho_\infty}$, *a* $\mathtt{N}$, $C_{h_\infty},\ \mu_{s,\infty},\ \mu_{u,\infty},\ \mu_{c,+,\infty},\ \mu_{c,-,\infty}$-*whiskered analytic splitting (with respect to* $A_{K_\infty,\beta_\infty}$*) over* $\mathbb{T}^d_{\rho_\infty}$.

---
[1] *Recall the operators* $\mathscr{I}$, $\mathscr{R}$ *which had been introduced in Subsections 4.1.1 and 4.1.3, respectively.*

- *An analytic vector bundle isomorphism* $\psi_\infty \in \mathcal{A}_{\rho_\infty,\ 0}^{\mathcal{L}\left(\mathbb{C}^c\times\mathbb{T}^d_{\rho_\infty},\ E^c_{S_\infty}\ \right)}$

such that

- $\mathscr{I}\,[K_\infty,\ \beta_\infty] \equiv 0.$

- $E^\sigma_{S_\infty} \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E^\sigma_0)$, $\sigma \in \{s, u, c\}$.

- *for each* $\{\sigma,\ \sigma',\ \sigma''\}$ *permutation of* $\{s, c, u\}$,

$$\mathscr{S}^\sigma[K_\infty,\ \beta_\infty,\ S_\infty^{\sigma,\ \sigma'},\ S_\infty^{\sigma,\sigma''}] \equiv 0$$

- $\mathscr{R}\,[\ K_\infty,\ \beta_\infty,\ S_\infty,\ U_\infty\ ] \equiv 0$

4.3 Outline of the proof of Theorem 1.

In this Section we present the main ideas in the proof of Theorem 1. Theorem 1 will be proved by a rapidly convergent method, as it is standard in KAM theory: Given approximate solutions of equations (26), we will produce a Newton method for all the equations.

The only non-standard aspect of our method is that the equations (26) have a very peculiar structure: The equation we are most interested in is the invariance equation of the torus, and the other equations can be considered as "preconditioners" whose main purpose is to solve the linearized equation for the invariance. This structure of the equations (26) is, in turn, reflected in a peculiar structure of the linear equations (26) which we will have to take advantage of (see Remark 14).

4.3.1 A Nash-Moser iterative procedure with floating parameters

In Theorem 1 we have set up a functional analysis problem: We wish to solve five functional equations:

$$\begin{cases} \mathscr{I}\,[\,K_\infty,\ \beta_\infty] \equiv 0 \\ \mathscr{S}^\sigma \left[\,K_\infty,\ \beta_\infty,\ S_\infty^{\sigma,\ \sigma'},\ S_\infty^{\sigma,\sigma''}\,\right] \equiv 0, \quad \sigma \in \{s,u,c\} \\ \mathscr{R}\,[\,K_\infty,\ \beta_\infty,\ S_\infty,\ U_\infty\,] \equiv 0 \end{cases} \tag{26}$$

We will find the solutions $K_\infty,\ \beta_\infty,\ S_\infty,\ U_\infty$ as limits of an iterative procedure (see Algorithm 1). In Section 5 we perform one step of the iteration. At step $\mathtt{k} \geq 0$, we have as data approximate solutions $K_\mathtt{k},\ \beta_\mathtt{k},\ S_\mathtt{k},\ U_\mathtt{k}$ that satisfy[2]:

[2]The initial approximate solutions (at step $\mathtt{k} = 0$) can be produced using formal expansions or as a result of a numerical computation: See [25, 26].

$$
\begin{cases}
\mathscr{I}\,[\,K_{\mathtt{k}},\ \beta_{\mathtt{k}}] := e_{\mathtt{k}} \\[2ex]
\mathscr{S}^{\sigma}\left[\ K_{\mathtt{k}},\ \beta_{\mathtt{k}},\quad S_{\mathtt{k}}^{\sigma,\ \sigma'},\ S_{\mathtt{k}}^{\sigma,\sigma''}\ \right] := e_{\mathscr{S}^{\sigma},\mathtt{k}},\quad \sigma\in\{s,u,c\} \\[2ex]
\mathscr{R}\,[\,K_{\mathtt{k}},\ \beta_{\mathtt{k}},\ S_{\mathtt{k}},\ U_{\mathtt{k}}\,] := e_{\mathscr{R},\mathtt{k}}
\end{cases}
\tag{27}
$$

where $E_{S_{\mathtt{k}}}$ is a $C_{h,\mathtt{k}}, \mu_{s,\mathtt{k}}, \mu_{u,\mathtt{k}}, \mu_{c,\mathtt{k}}$-Whiskered splitting (with respect to $A_{K_{\mathtt{k}},\beta_{\mathtt{k}}}$). We will also need to consider certain *condition numbers* which can be computed from $K_{\mathtt{k}},\ \beta_{\mathtt{k}},\ S_{\mathtt{k}},\ U_{\mathtt{k}}$. The condition numbers are associated to a non-degeneracy condition, as well as certain geometric conditions, which must be satisfied in order to perform each step of the iteration.

Newton's method consists in finding additive corrections $\Delta_{\mathtt{k}} \in \mathcal{A}^{\mathbb{C}^n}_{\rho_{\mathtt{k}+1}},\ \delta_{\mathtt{k}} \in \mathbb{C}^c,\ \chi^{\sigma}_{\mathtt{k}} \in \mathcal{A}^{\mathcal{L}(E_0^{\sigma},\ E_0^{\sigma'\oplus\sigma''})}_{\rho_{\mathtt{k}+1},\ 0},\ U_{\mathtt{k}} \in \mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho_{\mathtt{k}+1}}$ such that, if we set as output of step $\mathtt{k}$ the improved solutions:

$$
\begin{aligned}
&K_{\mathtt{k}+1} := K_{\mathtt{k}} + \Delta_{\mathtt{k}} \\
&\beta_{\mathtt{k}+1} := \beta_{\mathtt{k}} + \delta_{\mathtt{k}} \\
&S^{\sigma}_{\mathtt{k}+1} := S^{\sigma}_{\mathtt{k}} + \chi^{\sigma}_{\mathtt{k}},\quad \sigma\in\{s,u,c\} \\
&U_{\mathtt{k}+1} := U_{\mathtt{k}} + W_{\mathtt{k}}
\end{aligned}
\tag{28}
$$

then the norms of the updated errors: $|\ e_{\mathtt{k+1}}\ |_{\rho_{\mathtt{k}+1}},\ |\ e^{\sigma}_{\mathscr{S},\mathtt{k+1}}\ |_{\rho_{\mathtt{k}+1}},\ |\ e_{\mathscr{R},\mathtt{k+1}}\ |_{\rho_{\mathtt{k}+1}}$ admit tame estimates in the sense of Nash-Moser. Note that the domain of the improved solutions changes with the step of the iteration $\mathtt{k}$: We will set the following domain losses:

$$
\zeta_{\mathtt{k}} := \frac{\rho_0}{2^{\mathtt{k}+2}}\ ,\qquad \rho_{\mathtt{k}+1} := \rho_{\mathtt{k}} - \zeta_{\mathtt{k}}
$$

We must also keep track of the quality constants, which will deteriorate after each step.

4.3.2 The Newton equations for all unknowns

Consider the Taylor expansions up to first order:

$$
\begin{cases}
e_{\mathrm{k}+1} \;=\; e_{\mathrm{k}} + D\mathscr{I}\left[(K_{\mathrm{k}},\beta_{\mathrm{k}});(\Delta_{\mathrm{k}},\ \delta_{\mathrm{k}})\right] \\
\qquad\qquad\qquad\qquad + \mathcal{E}_2(\mathscr{I})\left[(K_{\mathrm{k}},\beta_{\mathrm{k}})\ ;\ \ (\Delta_{\mathrm{k}},\delta_{\mathrm{k}})\right] \\
e_{\mathscr{S}^\sigma,\mathrm{k}+1} \;=\; e_{\mathscr{S}^\sigma,\mathrm{k}} + D\mathscr{S}^\sigma\left[(K_{\mathrm{k}},\ \beta_{\mathrm{k}},\ \ S_{\mathrm{k}}^{\sigma,\ \sigma'},\ S_{\mathrm{k}}^{\sigma,\sigma''})\ ;\ \ (\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}}^{\sigma,\ \sigma'},\chi_{\mathrm{k}}^{\sigma,\ \sigma''})\right] \\
\qquad\qquad + \mathcal{E}_2(\mathscr{S}^\sigma)\left[(K_{\mathrm{k}},\beta_{\mathrm{k}},S_{\mathrm{k}}^{\sigma,\sigma'},S_{\mathrm{k}}^{\sigma,\sigma''})\ ;\ \ (\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}}^{\sigma,\ \sigma'},\chi_{\mathrm{k}}^{\sigma,\ \sigma''})\right] \\
\qquad\qquad\qquad\qquad\qquad\qquad ,\ \sigma\in\{s,u,c\} \\
e_{\mathscr{R},\mathrm{k}+1} \;=\; e_{\mathscr{R},\mathrm{k}} + D\mathscr{R}\left[(K_{\mathrm{k}},\beta_{\mathrm{k}},S_{\mathrm{k}},U_{\mathrm{k}})\ ;\ \ (\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}},W_{\mathrm{k}})\right] \\
\qquad\qquad + \mathcal{E}_2(\mathscr{R})\left[(K_{\mathrm{k}},\beta_{\mathrm{k}},S_{\mathrm{k}},U_{\mathrm{k}})\ ;\ \ (\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}},W_{\mathrm{k}})\right]
\end{cases}
$$

where the terms

$$\mathcal{E}_2(\mathscr{I})\left[(K_{\mathrm{k}},\beta_{\mathrm{k}})\ ;\ \ (\Delta_{\mathrm{k}},\delta_{\mathrm{k}})\right]$$

$$\mathcal{E}_2(\mathscr{S}^\sigma)\left[(K_{\mathrm{k}},\beta_{\mathrm{k}},S_{\mathrm{k}}^{\sigma,\ \sigma'},S_{\mathrm{k}}^{\sigma,\sigma''})\ ;\ \ (\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}}^{\sigma,\ \sigma'},\chi_{\mathrm{k}}^{\sigma,\ \sigma''})\right]\ ,\quad \sigma\in\{s,u,c\}$$

$$\mathcal{E}_2(\mathscr{R})\left[(K_{\mathrm{k}},\beta_{\mathrm{k}},S_{\mathrm{k}},U_{\mathrm{k}})\ ;\ \ (\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}},W_{\mathrm{k}})\right]$$

are of higher order.

Note that, if the corrections $(\Delta_{\mathrm{k}},\ \delta_{\mathrm{k}},\chi_{\mathrm{k}},W_{\mathrm{k}})$ satisfy following system of Newton equations

$$
\begin{cases}
D\mathscr{I}\left[(K_{\mathrm{k}},\beta_{\mathrm{k}})\ ;\quad (\Delta_{\mathrm{k}},\ \delta_{\mathrm{k}})\right] = -e_{\mathrm{k}} \\
D\mathscr{S}^{\sigma}\left[(K_{\mathrm{k}},\ \beta_{\mathrm{k}},\quad S_{\mathrm{k}}^{\sigma,\ \sigma'},\ S_{\mathrm{k}}^{\sigma,\sigma''})\ ;\quad (\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}}^{\sigma,\ \sigma'},\chi_{\mathrm{k}}^{\sigma,\ \sigma''})\right] = -e_{\mathscr{S}^{\sigma},\mathrm{k}}, \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma\in\{s,u,c\} \\
D\mathscr{R}\left[(K_{\mathrm{k}},\beta_{\mathrm{k}},S_{\mathrm{k}},U_{\mathrm{k}})\ ;\quad (\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}},W_{\mathrm{k}})\right] = -e_{\mathscr{R},\mathrm{k}}
\end{cases}
\tag{29}
$$

then, it holds that

$$
\begin{cases}
e_{\mathrm{k}+1} = \mathcal{E}_2(\mathscr{I})\left[(K_{\mathrm{k}},\beta_{\mathrm{k}});(\Delta_{\mathrm{k}},\delta_{\mathrm{k}})\right] \\
e_{\mathscr{S}^{\sigma},\mathrm{k}+1} = \mathcal{E}_2(\mathscr{S}^{\sigma})\left[(K_{\mathrm{k}},\beta_{\mathrm{k}},S_{\mathrm{k}}^{\sigma,\ \sigma'},\ S_{\mathrm{k}}^{\sigma,\sigma''});(\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}}^{\sigma,\ \sigma'},\chi_{\mathrm{k}}^{\sigma,\ \sigma''})\right]\ , \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma\in\{s,u,c\} \\
e_{\mathscr{R},\mathrm{k}+1} = \mathcal{E}_2(\mathscr{R})\left[(K_{\mathrm{k}},\beta_{\mathrm{k}},S_{\mathrm{k}},U_{\mathrm{k}});(\Delta_{\mathrm{k}},\delta_{\mathrm{k}},\chi_{\mathrm{k}},W_{\mathrm{k}})\right]
\end{cases}
$$

Remark 14. *Note that the system* (29) *has an approximately upper triangular format: The second, third and fourth equations on the functional $\mathscr{S}^{\sigma}$ involve one more variable than the first equation on $\mathscr{I}$. The last equation on the functional $\mathscr{R}$ involves one more variable than the equations on the functional $\mathscr{S}^{\sigma}$. This suggest that we solve it in order, from the top to the bottom.*

This heuristic is, unfortunately, not completely accurate. Note that system (29) *involves 5 equations and 6 unknowns. The saving grace is that, as we will see in Subsection 5.5, the Newton method for the Reducibility equation involves solving a cohomology equation with an obstruction (see assumption* (72) *in Lemma 5). The additive correction for the parameter, δ_{k}, is determined in order to satisfy the obstruction. Hence, the Newton method for the Reducibility equation determines two unknowns.*

One complication is that the reducibility equation is the last equation in (29)*. Thus, the cohomology equation, whose obstruction determines $\delta_{\mathtt{k}}$, involves the solutions of the previous equations in* (29)*. The natural way out is to proceed as in the substitution method of elementary linear algebra. We will find parameterized (by $\delta_{\mathtt{k}}$) solutions for the first equations, noting that the unknown $\delta_{\mathtt{k}}$ is now thought of as a parameter. Then, finally, the last equation will determine also $\delta_{\mathtt{k}}$. Substituting $\delta_{\mathtt{k}}$ in the parameterized solutions for the other equations we obtain a solution of all the equations.*

The expert reader will note that the paper [46] implicitly faces a similar difficulty. There are equations for the invariance and equations for the reduction to constants of the linear part which, in principle, have the same structure. The method of changing variables leads, however, to a system of equations which is diagonal. Then, all the parameters $\delta_{\mathtt{k}}$ do not affect the other equations.

In the language of Nash-Moser implicit function theory (see [67], [22], [29]) we will be producing *approximate right-inverses* for the invariance equations (27) but, to allow for the floating parameters, the approximate inverses will operate on affine families and produce affine families.

More precisely, the approximate right inverse of $\mathscr{I}$ will start from a functional error, $\mathscr{I}[K, \beta]$, and produce an affine function of δ. The approximate inverse of $\mathscr{S}$ will take as input an affine function and produce another affine function. Finally, the approximate right inverse of $\mathscr{R}$ will take an affine function and produce both a correction δ for the parameter and a correction for the cocycle U.

4.4 The algorithm

In this Section we present the pseudocode (Algorithm 1) for the step in the Nash-Moser iterative procedure.

Algorithm 1 Nash-Moser Step

global variables
    $f,\ K,\ \beta,\ E_0,\ \psi_0,\ S,\ U$
end global variables

if $K\left(\mathbb{T}^d_\rho\right) \not\subset \mathcal{D}_{2\rho_0}\left(K_0\right)$ **then** STOP

$e \leftarrow \mathscr{I}\left[K, \beta\right]$
$\Delta \leftarrow$ IMPROVEMENT$\mathscr{I}(e;\ \Delta)$ ▷ The affine function IMPROVEMENT$\mathscr{I}$ is defined in Algorithm 2 below. The output Δ is an affine function of δ.

if $\exists\ \sigma \in \{s, u, c\}\ :\ E^\sigma_S \notin \mathcal{V}_{\eta_0^{\mathscr{S}}}\left(E^\sigma_0\right)$ **then** STOP

for $\sigma \in \{s, u, c\}$ **do**
    $e^\sigma_{\mathscr{S}} \leftarrow \mathscr{S}^\sigma\left[K, \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''}\right]$
    for $\sigma' \neq \sigma$ **do**
        $\chi^{\sigma,\sigma'} \leftarrow$ IMPROVEMENT$\mathscr{S}^{\sigma,\sigma'}(e^\sigma_{\mathscr{S}}, \Delta;\ \chi^{\sigma,\sigma'})$ ▷ The affine IMPROVEMENT$\mathscr{S}^{\sigma,\sigma'}$ is defined in Algorithm 3 below. The output χ is an affine function of δ.

$\chi \leftarrow \prod_{\sigma\neq\sigma'} \chi^{\sigma,\sigma'}$

$e_{\mathscr{R}} \leftarrow \mathscr{R}\left[K, \beta, S, U\right]$

if $\min_{1\leq j\leq c}\left|\ \widehat{\left[\ \left(\ U^{(-1)} \circ T_\omega\ \eta^{\urcorner}_{\mathscr{R}}\ \right)\ \right]_{jj}}(0)\ \right| = 0$ **then** STOP
▷ The function $\eta^{\urcorner}_{\mathscr{R}}$ is defined in Lemma 9.

$(W, \gamma) \leftarrow$ IMPROVEMENT$\mathscr{R}(e_{\mathscr{R}}, \Delta, \chi;\ \gamma, W)$ ▷ The function IMPROVEMENT$\mathscr{S}$ is defined in Algorithm 3 below.

$K \leftarrow K + \Delta[\gamma]$
$\beta \leftarrow \beta + \gamma$
$S \leftarrow S + \chi[\gamma]$
$U \leftarrow U + W$

Algorithm 2 Pseudocode for function IMPROVEMENT$\mathscr{I}$

function IMPROVEMENT$\mathscr{I}(e;\ \Delta)$ ▷ The function IMPROVEMENT$\mathscr{I}$ takes as input e, the error of the invariance equation for K and produces as output an affine function Δ, to be used as input for functions IMPROVEMENT$\mathscr{S}^{\sigma,\sigma'}$ and IMPROVEMENT$\mathscr{R}$ below.

$\mathtt{Cte}(\eta) \leftarrow e$
$\mathtt{Lin}(\eta) \leftarrow \partial_\beta f_\beta \circ K$

for $\sigma \in \{s, u, c\}$ **do**
$\eta_\sigma \leftarrow \Pi_S^\sigma \circ T_\omega \ \ \eta$
$\Delta_\sigma \leftarrow \widetilde{\Gamma}_\sigma(\eta_\sigma)$ ▷ The operator $\widetilde{\Gamma}_\sigma$ is defined in Remark 16, Subsection 5.1, for $\sigma \in \{s, u\}$ and in Remark 21, Subsection 5.3, for $\sigma = c$.

$\Delta \leftarrow \Delta_s + \Delta_u + \Delta_c$

return Δ ▷ Note that Δ is an affine function of δ.

Algorithm 3 Pseudocode for function IMPROVEMENT$\mathscr{S}^{\sigma,\sigma'}$

function IMPROVEMENT$\mathscr{S}^{\sigma,\sigma'}(e^\sigma_{\mathscr{S}}, \Delta;\ \chi^{\sigma,\sigma'})$ ▷ The function IMPROVEMENT$\mathscr{S}^{\sigma,\sigma'}$ takes as inputs Δ, computed previously in IMPROVEMENT$\mathscr{I}$ and $e_{\mathscr{S}}$, the error of the invariance equation for the splitting E_S. IMPROVEMENT$\mathscr{S}^{\sigma,\sigma'}$ then produces as output affine functions $\chi^{\sigma,\sigma'}$, to be used as input for function IMPROVEMENT$\mathscr{R}$ below.

$$
\begin{aligned}
e^{\sigma,\sigma'}_{\mathscr{S}} &\leftarrow \Pi^{\sigma'}_S \circ T_\omega\ e^\sigma_{\mathscr{S}} \\
\mathtt{Cte}(\eta^{\sigma,\sigma'}_{\mathscr{S}}) &\leftarrow e^{\sigma,\sigma'}_{\mathscr{S}} + DA^{\sigma',\sigma'}_{K,\beta}\mathtt{Cte}(\Delta) S^{\sigma,\sigma'} \\
&\qquad - S^{\sigma,\sigma'} \circ T_\omega DA^{\sigma,\sigma}_{K,\beta}\mathtt{Cte}(\Delta)
\end{aligned}
$$

$$
\begin{aligned}
\mathtt{Lin}(\eta^{\sigma,\sigma'}_{\mathscr{S}}) &\leftarrow \left[DA^{\sigma',\sigma'}_{K,\beta}\mathtt{Lin}(\Delta) + \partial_\beta A^{\sigma',\sigma'}_{K,\beta}\right] S^{\sigma,\sigma'} \\
&\quad - S^{\sigma,\sigma'} \circ T_\omega \left[DA^{\sigma,\sigma}_{K,\beta}\mathtt{Lin}(\Delta) + \partial_\beta A^{\sigma,\sigma}_{K,\beta}\right]
\end{aligned}
$$

$\chi^{\sigma,\sigma'} \leftarrow \widetilde{\Gamma}_{\mathscr{S}^{\sigma,\sigma'}}(\eta^{\sigma,\sigma'}_{\mathscr{S}})$ ▷ The operator $\widetilde{\Gamma}_{\mathscr{S}^{\sigma,\sigma'}}$ is defined in Remark 21, Subsection 5.4.

return $\chi^{\sigma,\sigma'}$ ▷ Note that $\chi^{\sigma,\sigma'}$ is an affine function of δ.

Algorithm 4 Pseudocode for function IMPROVEMENT$\mathscr{R}$

function IMPROVEMENT$\mathscr{R}(e_{\mathscr{R}}, \Delta, \chi;\ \gamma, W)$ ▷ The function IMPROVEMENT$\mathscr{R}$ takes as inputs Δ, computed previously in IMPROVEMENT$\mathscr{I}$, χ, computed previously in IMPROVEMENT$\mathscr{S}^{\sigma,\sigma'}$ and $e_{\mathscr{R}}$, the error of the reducibility equation for the bundle of frames U. IMPROVEMENT$\mathscr{R}$ then produces as outputs both a correction W for U and a correction γ for β. The correction of the parameter, γ, is then used to compute $\Delta[\gamma]$, the correction for K and $\chi[\gamma]$, the correction for S.

$$\begin{aligned} Z_{\Xi} \leftarrow & \chi^{s,c}\ \Pi_0^s \begin{pmatrix} Id_{E_0^s} & S^{u,s} \\ S^{s,u} & Id_{E_0^u} \end{pmatrix}^{-1} + \ \chi^{u,c}\ \Pi_0^u \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1} \\ & + \ (S^{s,c}\ \Pi_0^s + S^{u,c}\ \Pi_0^u) \ \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \chi^{u,s} \\ \chi^{s,u} & 0 \end{pmatrix} \\ & \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1} \end{aligned}$$

$$\begin{aligned} \eta_{\mathscr{R}}^{\daleth} \leftarrow & \ \left(\psi_0^{-1}\ \ \Pi_{\daleth}^c\right) \circ T_\omega\ \ [DA_{K,\beta}\ \Delta + \partial_\beta A_{K,\beta}\ \delta]\quad \psi_0\ U \\ & - \left(\psi_0^{-1}\ \Pi_{\daleth}^c \begin{pmatrix} 0 & Z_{\Xi} \\ 0 & 0 \end{pmatrix} \Pi_{\daleth}^c\right) \circ T_\omega\ \ A_{K,\beta}\ \ \psi_0\ U \end{aligned}$$

$(\gamma, W) \leftarrow \Big(\ \widetilde{\gamma}_{\mathscr{R}}(\eta_{\mathscr{R}}^{\daleth} + e_{\mathscr{R}}),\ \widetilde{\Gamma}_{\mathscr{R},\xi}(\eta_{\mathscr{R}}^{\daleth} + e_{\mathscr{R}})\ \Big)$ ▷ The operators $\widetilde{\gamma}_{\mathscr{R}},\ \widetilde{\Gamma}_{\mathscr{R},\xi}$ are defined in Remark 23, Subsection 5.5.1.

return (γ, W)

4.5 Relations with previous results in the literature.

In this Section we briefly comment on some important results related to skew products (1) and to the proof of Theorem 1 that had been considered previously in the literature.

4.5.1 Fibered dynamics and strange non-chaotic attractors

The skew products (1) have attracted plenty of attention in the literature. A common theme is that the skew products (1) often present interesting dynamical properties, and already the case $n = d = 1$ has been a source of examples: In [54], a fibered quadratic polynomial admitting two attractive invariant curves is constructed (this phenomenon is not present in the non-fibered case, since quadratic polynomials admit at most one attractive cycle). In [21], a skew product over an irrational translation was used to produce *Strange non-chaotic attractors* (SNAs), which are non-piecewise differentiable (strange) attractors with negative Lyapunov exponents. Earlier examples of SNAs appear in [42], [62], but the term SNA was coined in [21].

In the physics literature, quasi-periodically forced systems have been studied in many contexts: e.g. in quantum mechanical systems with quasi-periodic potentials, driven mechanical systems, electronic circuits, plasma systems, and neuronal membrane systems. The survey [56] reviews experiments and numerical studies that have been performed in order to obtain SNAs from quasi-periodically forced systems in each of these physical contexts. [56] also mentions applications in the area of secure communications.

More recently, there has been an effort to obtain new examples of families of skew products (1) exhibiting SNAs for positive measure sets of the parameter space. A fruitful approach has been to exploit the connections between the dynamics of some two-dimensional linear skew products (1) and the spectral properties of quasi-periodic Schrödinger operators. In [27] (see also [34]) this approach is applied to the *Harper map*

$$
\begin{aligned}
s_{a,E}\colon \mathbb{T}^1 \times \overline{\mathbb{R}} &\to \mathbb{T}^1 \times \overline{\mathbb{R}} \\
(\theta, x) &\mapsto \left(\theta + \omega,\ a\cos(2\pi\theta) - E - \frac{1}{x}\right).
\end{aligned}
\tag{30}
$$

For non linear skew products, the technique of parameter exclusion (introduced in [3] for the Hénon map, and in [66] and [4] for quasi-periodic $\mathrm{SL}(2,\mathbb{R})$ and Schrödinger cocycles, respectively) has been applied in [31], [30] to some families of quasiperiodically forced Arnold maps

$$
\begin{aligned}
f_{a,b,\tau}\colon \mathbb{T}^2 &\to \mathbb{T}^2 \\
(\theta, x) &\mapsto (\theta + \omega,\ x + \tau + a\sin(2\pi x)\ +\ bg(\theta)\ \mathrm{mod} 1)\,.
\end{aligned}
\tag{31}
$$

where $b,\ \tau \in \mathbb{R}$, $a \in [0,1]$ and $g : \mathbb{T}^1 \to \mathbb{R}$ is the forcing function.

[53] proposes the study of skew maps (1), in the case $m = n = 1$, as an intermediate step to obtain higher dimensional analogues of well-known results (see [9]) for the local dynamics around a fixed point of holomorphic germs in dimension one. The analogue of a fixed point of a holomorphic germ in dimension one is an invariant curve (in the case $d = 1$ it is standard to use the term curve instead of torus).

In [52] the persistence problem for invariant curves in one dimension is studied. This is a particular case of Theorem 1 in the case $m = n = 1$. [52] considers skew products with normal form

$$
\begin{aligned}
F_0\ :\ \mathbb{T}^1_\rho \times \mathbb{C} &\to \mathbb{T}^1_\rho \times \mathbb{C} \\
(\theta, z) &\mapsto \big(\theta + \omega,\ \big(\ e^{2\pi i\lambda} + \zeta(\theta, z)\ \big)\, z\big)
\end{aligned}
$$

where ζ is an analytic function vanishing up to second order at $z = 0$, and the pair $(\omega,\ \lambda)$ satisfy a joint number-theoretic condition[3].

[3] λ plays the role of the normal frequencies Λ in Theorem 1

Note that graph of $K_0(\theta) = 0,\ \theta \in \mathbb{T}^d_\rho$ is an invariant curve of F_0. [52] proves that the graph of K_0 is persistent in codimension 1 under suitable analytic pertubations of F_0. The proofs are very different from ours since they are based on transformation theory. We remark the following:

- [52] requires only a Brjuno-type condition on the pair (ω, λ). The Brjuno condition is weaker than the Melnikov conditions from Theorem 1. A Brjuno condition is optimal for some linearization problems [64, 65, 48].

- In [52] a *fibered rotation number* is introduced in order to detect the persistence under perturbation of the normal frequency λ. In [51], the fibered rotation number is used to describe the dynamics of the skew product around an invariant curve.

4.5.2 Lower dimensional elliptic tori

[46] presents a perturbation theory for quasi-periodic solutions of systems of differential equations with normal form

$$\begin{cases} \dot{x} \ = \ \omega + \epsilon\ f(x, y, \epsilon) + \lambda_0 \\ \dot{y} \ = \ \Lambda\ y + \epsilon\ g(x, y, \epsilon) + \mu_0 + M_0\ y \end{cases}$$

where $f : \mathbb{T}^d \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^d$, $g : \mathbb{T}^d \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are analytic, the tangent and normal frequencies $\omega \in \mathbb{T}^d$, $\Lambda \in \mathbb{R}^{n\times n}$ satisfy[4]

$$|\ k \cdot \omega \ + \ l \cdot \Lambda\ | \ \geq \ \nu\ (\ 1 + |\ k\ |^\tau\ )^{-1} \tag{32}$$

for all but a finite set of integer vectors $k \in \mathbb{Z}^d,\ l \in \mathbb{Z}^n$, and $\lambda_0 \in \mathbb{R}^d$, $\mu_0 \in \mathbb{R}^n$, $M_0 \in \mathbb{R}^{n\times n}$ are parameters satisfying

$$\Omega\ \mu_0 = 0\ , \quad \Omega\ M_0 = M_0\ \Omega$$

[4] The case $d = 1$ (*periodic solutions*) is much simpler as there are no small divisors. See [50]

[46] proves that there exist a positive radius of convergence $\rho_\infty > 0$, and unique analytic functions (called *counterterms*) $\lambda : (-\rho_\infty,\ \rho_\infty) \to \mathbb{R}^d$, $\mu : (-\rho_\infty,\ \rho_\infty) \to \mathbb{R}^n$, $M : (-\rho_\infty,\ \rho_\infty) \to \mathbb{R}^{n\times n}$ satisfying

$$\Omega\ \mu(\epsilon) = 0\ , \quad \Omega\ M(\epsilon) = M(\epsilon)\ \Omega\ , \quad \epsilon \in (-\rho_\infty,\ \rho_\infty)$$

such that the modified system

$$\begin{cases} \dot{x} &= \ \omega + \epsilon\ f(x,y,\epsilon) + \lambda(\epsilon) \\ \dot{y} &= \ \Lambda\ y + \epsilon\ g(x,y,\epsilon) + \mu(\epsilon) + M(\epsilon)\ y \end{cases}$$

admits a quasi-periodic solution with tangential frequency ω and normal frequency Λ.

We remark that [46] does not involve a non-degeneracy condition for the perturbed system because the family of perturbations considered in [46] admits additional parameters μ along the normal directions. Note also that the linearization $\dot{y} = \Lambda\ y$ must be trivializable in all the normal directions, and not only along the central bundle. The Newton scheme implemented in [46] does not require to take advantage of the upper triangular structure of the linearized equations.

[46] observes that if the system of differential equations has symmetries or preserves a geometric structure, then less parameters may be required. As an application, a perturbation theory for elliptic lower dimensional tori in Hamiltonian systems is developed as follows: On the phase space $\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n$, consider the hamiltonian system

$$\begin{cases} \dot{x} = \partial_y\ H \\ \dot{y} = -\partial_x\ H \end{cases} , \quad \begin{cases} \dot{u} = \partial_v\ H \\ \dot{v} = -\partial_u\ H \end{cases}$$

where H is the unperturbed hamiltonian

$$H(x,y,u,v)\ := \ \sum_{j=1}^{d} \omega_j\ y_j + \frac{1}{2} \sum_{j=1}^{n} \Lambda_j \left(u_j^2 + v_j^2\right),$$

Let $\mathcal{X}_H$ be the flow associated to H. The torus $\mathfrak{K}_0 := (x, 0, 0, 0)$, $x \in \mathbb{T}^d$, is $\mathcal{X}_H$-invariant with tangential frequencies ω and normal frequencies $(\Lambda, -\Lambda)$. The question is then, given a suitable perturbation $H + P$ of the hamiltonian H, to find an $\mathcal{X}_{H+P}$-invariant d-dimensional torus $\mathfrak{K}_\infty$ *with the same tangential frequencies* ω *as* $\mathfrak{K}_0$ near $\mathfrak{K}_0$. The case $n = 0$ corresponds to the *maximal tori*, famously studied in [36, 37, 2]. For maximal tori, if a non-degeneracy condition is satisfied, there are sufficiently many parameters to restore the tangential frequency ω.

For lower dimensional ($n \geq 1$) stable tori, one encounters the *lack of parameters* problem (see [60]): The normal frequencies will, in general, be functions $\Lambda(\omega)$ of the tangential frequencies. There are not enough parameters available in the system in order to fix *both* the tangential and the normal frequencies, and thus ensuring condition (32) is difficult. [46] treats the case $n = 1$ (called *next to maximal* tori), but the general case can not be dealt with the technique of [46], where all the frequencies remain fixed throughout the iterative procedure. The breakthrough came in [13] (see also [55]), were a perturbation theory is developed under the assumption that a non-degeneracy condition for the normal frequencies is satisfied. A key difference of [13] with [46] is that in [13] the normal frequencies are allowed to vary during the iterative procedure.

Non-degeneracy conditions as the one in Theorem 1 are standard in the literature. However, degenerate systems are encountered frequently in applications, e.g. existence of quasi-periodic motions in planetary systems [17, 1] . Reference [61] (see also [5, 60, 7]) constructs Whitney-smooth families of invariant lower-dimensional tori over Cantor sets, assuming a very weak non-degeneracy condition. The method used in [61], presented already by Michael Herman in 1990 (Herman did not publish his work, but see [63]), consists of adding sufficiently many external parameters to remove all degeneracies. Then, to the new non-degenerate systems, the theory of unfoldings of invariant tori in [6] is applied. One thus obtains families of invariant lower-dimensional tori for the extended non-degenerate system. Finally, to find invariant tori for the original degenerate system, classical results [49] in the metric theory of Diophantine approximations can be applied if a Pjartly-type non-degeneracy condition is satisfied.

4.5.3 The parameterization method

The papers [25, 23, 24] consider the case of hyperbolic systems (the center bundle is trivial). The fact that the numerical methods work also in the elliptic case is observed empirically. This paper provides some justification. Other works on lower dimensional elliptic tori are [41, 32, 35, 33].

CHAPTER V

SOLUTIONS AND APPROXIMATE SOLUTIONS FOR THE LINEARIZED EQUATIONS.

In this Chapter we will solve (or solve approximately) several types of linear equations, which will be used in Section 6 to find corrections for the invariance and reducibility functionals. These corrections are the output of each step of the iterative procedure outlined in Algorithm 1.

5.1 Solution of the linearized invariance equation in the stable and unstable directions

We now prove Lemma 1, where we construct solutions Δ_σ of the linear invariance equations (33) in the stable and unstable directions, which are of the form:

$$A^{\sigma;\sigma}_{K,\ \beta}[S]\ \Delta_\sigma - \Delta_\sigma \circ T_\omega = -\eta_\sigma \tag{33}$$

where $\sigma \in \{s, u\}$ and η_σ is an affine function of δ (as in (34)). The solutions Δ_σ constructed in Lemma 1 are affine functions of δ. Recall from Remark 14 that Δ_σ is a solution of (33) in the sense that, given any $\delta \in \mathbb{C}^c$, it holds that

$$A^{\sigma;\sigma}_{K,\ \beta}[S]\ \Delta_\sigma[\delta] - \Delta_\sigma[\delta] \circ T_\omega = -\eta_\sigma[\delta]$$

Estimates for the solution Δ_σ in terms of the right-hand side of (33), η_σ, are given in (35) .

In Section 6.1 we will specify the right-hand side term η_σ in the linear equation (33) which we will use in the iterative procedure [1].

[1]Since η_σ is not yet specified, the statement of Lemma 1 holds for η_σ any affine function of δ as in (34).

Lemma 1. *Let* $K \in \mathcal{A}_{\rho}^{\mathbb{C}^c}$, $\beta \in \mathbb{C}^c$, $S^\sigma \in \mathcal{A}_{\rho,0}^{\mathcal{L}\left(E_0^\sigma, E_0^{\sigma'\oplus\sigma''}\right)}$ *for* $\sigma \in \{s,u,c\}$. *Assume that the splitting* E_S *is* $\mathtt{N}$, C_h, $\mu_{K,\beta}^{S,s}$, $\mu_{K,\beta}^{S,u}$, $\mu_{K,\beta}^{S,c,+}$, $\mu_{K,\beta}^{S,c,-}$*- whiskered with respect to* $A_{K,\beta}$ *and, for* $\sigma \in \{s,u\}$, *let*

$$\begin{aligned} \eta_\sigma \ : \ \mathbb{C}^c &\to \mathcal{A}_{\rho,\omega}^{E_S^\sigma} \\ \delta &\mapsto \eta_\sigma[\delta]. \end{aligned} \tag{34}$$

be an affine function of δ.

Then, there exist solutions of (33)

$$\begin{aligned} \Delta_\sigma \ : \ \mathbb{C}^c &\to \mathcal{A}_{\rho,0}^{E_S^\sigma} \\ \delta &\mapsto \Delta_\sigma[\delta]. \end{aligned}$$

defined by the series (38), *such that:*

- Δ_σ *are affine functions of* δ.
- *Denote by* $\mathtt{Cte}(\Delta_\sigma)$, $\mathtt{Lin}(\Delta_\sigma)$ *the constant and linear parts of* Δ_σ, *respectively. The following estimates hold:*

$$\begin{aligned} |\ \mathtt{Cte}(\Delta_\sigma)\ |_\rho &\leq \frac{C}{1-\mu_{K,\beta}^{S,\sigma}}\ |\ \mathtt{Cte}(\eta_\sigma)\ |_\rho\,, \\ |\ \mathtt{Lin}(\Delta_\sigma)\ |_\rho &\leq \frac{C}{1-\mu_{K,\beta}^{S,\sigma}}\ |\ \mathtt{Lin}(\eta_\sigma)\ |_\rho \end{aligned} \tag{35}$$

Furthermore, given any $\delta \in \mathbb{C}^c$, $\Delta_\sigma[\delta]$ *is the unique solution in* $\mathcal{A}_{\rho,0}^{E_S^\sigma}$ *of the equation*

$$A_{K,\ \beta}^{\sigma,\sigma}[S]\ \Delta_\sigma[\delta] - \Delta_\sigma[\delta]\circ T_\omega = -\eta_\sigma[\delta] \tag{36}$$

Proof. The solutions of (33) must satisfy:

$$\begin{aligned} \Delta_s &= \left(\ A_{K,\beta}^{s,s}[S]\ \Delta_s + \eta_s\ \right)\circ T_{-\omega} \\ \Delta_u &= \left(\ A_{K,\beta}^{u,u}[S]\ \right)^{(-1)}\left(\ \Delta_u\circ T_\omega - \eta_u\ \right) \end{aligned} \tag{37}$$

Iterating (37), we obtain candidate expressions for the solutions of (33):

$$
\begin{aligned}
\Delta_s &:= \sum_{k=0}^{\infty} \left(A^{s,s}_{K,\beta}[S] \right)^{(k)} \circ T_{-k\omega} \quad \eta_s \quad \circ T_{-(k+1)\omega} \\
\Delta_u &:= \sum_{k=0}^{\infty} \left(A^{u,u}_{K,\beta}[S] \right)^{(-k)} \circ T_{k\omega} \quad \eta_u \circ T_{k\omega}
\end{aligned}
\tag{38}
$$

The general term of the sums admit the estimates

$$
\begin{aligned}
&\left| \left(A^{s,s}_{K,\beta}[S] \right)^{(k)} \circ T_{-k\omega} \quad \eta_s \circ T_{-(k+1)\omega} \right|_\rho \\
&\le \left| \left(A^{s,s}_{K,\beta}[S] \right)^{(k)} \right|_\rho \quad |\ \eta_s\ |_\rho \\
&\le C_h \left(\mu^{S,s}_{K,\beta} \right)^k |\ \eta_s\ |_\rho \ ,
\end{aligned}
$$

similarly,

$$
\left| \left(A^{u,u}_{K,\beta}[S] \right)^{(-k)} \circ T_{k\omega} \quad \eta_u \circ T_{k\omega} \right|_\rho \le C_h \left(\mu^{S,u}_{K,\beta} \right)^k |\ \eta_u\ |_\rho
$$

and so

$$
\begin{aligned}
|\ \Delta_s\ |_\rho &\le \sum_{k=0}^{\infty} \left| \left(A^{s,s}_{K,\beta}[S] \right)^{(k)} \circ T_{-k\omega} \quad \eta_s \circ T_{-(k+1)\omega} \right|_\rho \\
&\le \sum_{k=0}^{\infty} C_h \left(\mu^{S,s}_{K,\beta} \right)^k |\ \eta_s\ |_\rho \\
&\le \frac{C}{1-\mu^{S,s}_{K,\beta}} |\ \eta_s\ |_\rho \ , \\
|\ \Delta_u\ |_\rho &\le \frac{C}{1-\mu^{S,u}_{K,\beta}} |\ \eta_u\ |_\rho
\end{aligned}
\tag{39}
$$

(we have used the fact that $\mu^{S,u}_{K,\beta} < 1$, $\mu^{S,s}_{K,\beta} < 1$). Hence, the series (38) converge absolutely in $\mathcal{A}_\rho^{E_S^\sigma}$. The absolute convergence of series (38) justifies the rearrangements needed to show that the series (38) converge absolutely to parameterized solutions of the parameterized linear equations (33).

Remark 15. *Note that Δ_σ depends linearly on η_σ, which is an affine function of δ. Hence, Δ_σ is an affine function of δ. Denote* $\texttt{Cte}(\Delta_\sigma)$, $\texttt{Lin}(\Delta_\sigma)$ *the constant and* *linear parts of Δ_σ, respectively. Clearly, we have*

$$
\begin{aligned}
\mathtt{Cte}(\Delta_s) &= \sum_{k=0}^{\infty} \left(\ A^{s,s}_{K,\beta}[S]\ \right)^{(k)} \circ T_{-k\omega}\ \ \mathtt{Cte}(\eta_s)\ \circ T_{-(k+1)\omega} \\
\mathtt{Lin}(\Delta_u) &= \sum_{k=0}^{\infty} \left(\ A^{u,u}_{K,\beta}[S]\ \right)^{(-k)} \circ T_{k\omega}\ \ \mathtt{Lin}(\eta_u) \circ T_{k\omega}
\end{aligned} \tag{40}
$$

Estimates (35) follow by applying estimates analogous to (39) to the expressions (40).

Finally, given any $\delta \in \mathbb{C}^c$, let $\Delta_\sigma[\delta]$, $\widetilde{\Delta}_\sigma[\delta] \in \mathcal{A}^{E^\sigma_S}_{\rho,0}$ be solutions of (36), and $N \in \mathbb{N}$ arbitrarily large. Iterating (37) N times, we see that

$$
\begin{aligned}
\Delta_s[\delta] - \widetilde{\Delta}_s[\delta] &= \left(\ A^{s,s}_{K,\beta}[S]\ \right)^{(N)} \circ T_{-N\omega}\ \ \left(\ \Delta_s[\delta] - \widetilde{\Delta}_s[\delta]\ \right) \circ T_{-N\omega} \\
\Delta_u[\delta] - \widetilde{\Delta}_u[\delta] &= \left(\ A^{u,u}_{K,\beta}[S]\ \right)^{(-N)} \circ T_{N\omega}\ \ \left(\ \Delta_u[\delta] - \widetilde{\Delta}_u[\delta]\ \right) \circ T_{(N+1)\omega}
\end{aligned} \tag{41}
$$

Hence, for any $N \in \mathbb{N}$, we have

$$
\begin{aligned}
\left|\ \Delta_s[\delta] - \widetilde{\Delta}_s[\delta]\ \right|_\rho &\le C_h \left(\mu^{S,s}_{K,\beta}\right)^N \left|\ \Delta_s[\delta] - \widetilde{\Delta}_s[\delta]\ \right|_\rho \\
\left|\ \Delta_u[\delta] - \widetilde{\Delta}_u[\delta]\ \right|_\rho &\le C_h \left(\mu^{S,u}_{K,\beta}\right)^N \left|\ \Delta_u[\delta] - \widetilde{\Delta}_u[\delta]\ \right|_\rho
\end{aligned} \tag{42}
$$

as $\left(\mu^{S,s}_{K,\beta}\right)^N \xrightarrow{N\to\infty} 0$, $\quad \left(\mu^{S,s}_{K,\beta}\right)^N \xrightarrow{N\to\infty} 0$, it follows that $\Delta_\sigma[\delta] = \widetilde{\Delta}_\sigma[\delta]$. This establishes the uniqueness of the solutions of (33) in $\mathcal{A}^{E^\sigma_S}_{\rho,0}$. □

Remark 16. *It will be typographically convenient for future applications to introduce the following notation: Let $\widetilde{\Gamma}_{\mathscr{S}^\sigma}$, $\sigma \in \{s, u\}$, be the operators acting on affine functions from $\mathbb{C}^c$ taking values in $\mathcal{A}^{E^\sigma_S}_{\rho,0}$ defined by*

$$
\widetilde{\Gamma}_{\mathscr{S}^\sigma}\left(\ \eta_\sigma\ \right)[\delta] := \Delta_\sigma[\delta]
$$

where $\Delta_\sigma[\delta]$ is the unique solution in $\mathcal{A}^{E^\sigma_S}_{\rho,0}$ of equation (33) constructed in Lemma 1.

5.2 Solution of a cohomology equation associated to the linear invariance equation in the central directions

In this Section we study the cohomology equation (45). In Lemma 2 we obtain solutions V of (45), which depend affinely on the parameter δ. Estimates for the solution in terms of the right-hand side of (45) are given in (46). Estimates (46) are well known in the literature: We reproduce them here for completeness.

In Section 5.3 we will obtain a quasi-Newton equation for the projection onto E^c of the correction Δ for $\mathscr{I}$. The cohomology equation (45) is equivalent to the quasi-Newton equation: As we will see in Section 5.3, equation (45) is a trivialized version of the quasi-Newton equation, obtained using ψ_0, the trivialization of the central bundle E^c_0.

In Section 6.1 we will specify the right-hand side term η in (45) which will be used in the iterative procedure [2].

Lemma 2. *Let* $0 < \zeta < \rho$, $\omega \in \mathbb{T}^d$, $\Lambda = diag[\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_c] \in \mathbb{C}^{c\times c}$ *be such that*

$$\omega \in DC(\nu, \tau)\ ,$$

$$Spec\,(\Lambda) \in DC^{1st}_{\omega}(\nu, \tau).$$

Let

$$\begin{aligned} \eta\ :\ \mathbb{C}^c &\to \mathcal{A}^{\mathbb{C}^c}_{\rho,\omega} \\ \delta &\mapsto \eta[\delta]. \end{aligned} \tag{43}$$

be an affine function of δ*. Then, the function*

$$\begin{aligned} V\ :\ \mathbb{C}^c &\to \mathcal{A}^{\mathbb{C}^c}_{\rho-\zeta,\omega} \\ \delta &\mapsto V[\delta]. \end{aligned} \tag{44}$$

defined by (47) *is a solution of the cohomology equation:*

$$\Lambda V - V \circ T_\omega = -\eta \tag{45}$$

[2]The statement of Lemma 2 holds for η any affine function of δ as in (43).

Furthermore:

- *V is an affine function of δ.*
- *Denote by $\mathtt{Lin}(V)$, $\mathtt{Cte}(V)$ the linear and constant parts of V, respectively. The following estimates hold:*

$$
\begin{aligned}
&|\ \mathtt{Cte}(V)\ |_{\rho-\zeta} \le C\ \zeta^{-\tau}\ |\ \mathtt{Cte}(\eta)\ |_{\rho}, \\
&|\ \mathtt{Lin}(V)\ |_{\rho-\zeta} \le C\ \zeta^{-\tau}\ |\ \mathtt{Lin}(\eta)\ |_{\rho}
\end{aligned}
\tag{46}
$$

- *Given any $\delta \in \mathbb{C}^c$, $V[\delta]$ is the unique solution in $\mathcal{A}^{\mathbb{C}^c}_{\rho-\zeta,\omega}$ of the cohomology equation*

$$\Lambda V[\delta] - V[\delta] \circ T_\omega = -\eta[\delta]$$

Proof. Given any $\delta \in \mathbb{C}^c$, let

$$V[\delta] = (V_1[\delta],\ \ \dots,\ V_c[\delta]\ )^t, \quad \eta[\delta] = (\eta[\delta],\ \dots,\ \eta[\delta]\ )^t.$$

Using the Fourier method we see that (45) holds if, and only if,

$$\widehat{V_j}[\delta](k)\,(\ \lambda_j - \exp(2\pi i k\cdot\omega)) = -\widehat{\eta}[\delta](k)$$

for all $k \in \mathbb{Z}^d$, $1 \le j \le c$. Since $\Lambda \in DC^{1st}_{\omega}(\nu,\tau)$, the factor $\lambda_j - \exp(2\pi i k\cdot\omega)$ is not zero and we can define $V[\delta]$ formally as having Fourier coefficients

$$\widehat{V_j}[\delta](k) := -\frac{\widehat{\eta}[\delta](k)}{\lambda_j - \exp(2\pi i k\cdot\omega)} \tag{47}$$

Note that V, as defined in (47), is an affine function of δ. The estimates

$$|\ \mathtt{Cte}(V)\ |_{\rho-\zeta} \le C\ \zeta^{-(d+\tau)}\ |\ \mathtt{Cte}(\eta)\ |_{\rho},$$

$$|\ \mathtt{Lin}(V)\ |_{\rho-\zeta} \le C\ \zeta^{-(d+\tau)}\ |\ \mathtt{Lin}(\eta)\ |_{\rho}$$

are standard (see [12]). Note that the estimates (46) are better by a factor of ζ^{-d}. The improved estimates (46) had been obtained by Rüssmann (see [57]) for non-twisted cohomology equations (cohomology equations such that $\lambda_j = 1$, hence the small divisor problem involves only the basic frequency ω). Rüssmann's method has been adapted in [13] to treat twisted cohomology equations (such as (45)) as well.

The expression (47) for the Fourier coefficients of $V[\delta]$ defines V uniquely in $\mathcal{A}^{\mathbb{C}^c}_{\rho-\zeta,\omega}$

$\square$

5.3 *Approximate solution of the linear invariance equation in the central directions*

We now prove Lemma 3, where we construct *approximate* solutions Δ_c of the linear invariance equation, (48), in the central directions, which is of the form:

$$A^{c,c}_{K,\beta}[S]\ \Delta_c - \Delta_c \circ T_\omega = -\eta_c \tag{48}$$

where η_c is an affine function of δ (as in (55)). The approximate solution Δ_c that we construct in Lemma 3 is an affine function of δ. Recall from Remark 14 that Δ_c is an approximate solution of (48) in the sense that, given any $\delta \in \mathbb{C}^c$, it holds that

$$A^{c,c}_{K,\ \beta}[S]\ \Delta_c[\delta] - \Delta_c[\delta] \circ T_\omega = -\eta_c[\delta] + e^c_{qNwt}[\delta]$$

where e^c_{qNwt} is the error of the approximate solution, to be estimated in (58).

Estimates for the approximate solution Δ_c in terms of the right-hand side of (48), η_c, are given in (57).

In Section 6.1 we will specify the right-hand side term η_c in the linear equation (48) which will be used in the iterative procedure [3].

Recall that in Subsection 4.1.3 we had introduced the vector bundle isomorphism ψ_S and the reduced derivative cocycle along the central directions $\overline{A_{K,\beta}}[S]$. We begin by using ψ_S and the Approximate Reducibility identity

$$\overline{A^{c,c}_{K,\beta}}[S]\ U - U \circ T_\omega\ \Lambda = e_{\mathscr{R}} \tag{49}$$

to approximately reduce equation (48) into a constant coefficients equation, (54) which is furthermore diagonalized.

[3] The statement of Lemma 3 holds for η_c any affine function of δ as in (55).

5.3.0.1 *Trivialization of* (48) *and approximate reducibility to a diagonal, constant coefficients equation*

The vector bundle isomorphism ψ_S converts (48) to an equivalent vector equation in $\mathbb{C}^c$:

$$\overline{A^{c,c}_{K,\beta}}[S]\ \Delta_c^{\psi_S} - \Delta_c^{\psi_S} \circ T_\omega = -\eta_c^{\psi_S}\ , \tag{50}$$

where we have denoted the trivializations of the sections Δ_c, η_c as:

$$\Delta_c^{\psi_S} :=\ \psi_S^{-1}\ \Delta_c\ , \qquad\qquad \eta_c^{\psi_S} :=\ \psi_S^{-1}\ \eta_c$$

(here $\Delta_c^{\psi_S} \in \mathcal{A}_\rho^{\mathbb{C}^c}$, $\ \eta_c^{\psi_S} \in \mathcal{A}_\rho^{\mathbb{C}^c}$). We will find an approximate solution $\Delta_c^{\psi_S}$ for (50). Then we will set

$$\Delta_c = \psi_S\ \Delta_c^{\psi_S}$$

as the approximate solution of (48).

Solving for $\overline{A^{c,c}_{K,\beta}}[S]$ in the approximate Reducibility identity (49) we obtain:

$$\overline{A^{c,c}_{K,\beta}}[S] = [\ \ U\ \ \circ T_\omega\ \Lambda\ \ +\ \ e_{\mathscr{R}}\ ]\ \ U^{(-1)} \tag{51}$$

Introduce the change of variables:

$$\Delta_c^{\psi_S} = U\ V \tag{52}$$

Here, $V \in \mathcal{A}_\rho^{\mathbb{C}^c}$. We substitute the expression (51) for $\overline{A^{c,c}_{K,\beta}}[S]$ in (50), and use the change of variables (52) to rewrite (50) as:

$$[\ U \circ T_\omega\ \Lambda + e_{\mathscr{R}}\ ]\ \ V - (\ U\ V\ \ ) \circ T_\omega\ \ =\ \ -\eta_c^{\psi_S}$$

Hence, (50) is equivalent to

$$\Lambda\ V - V \circ T_\omega = -U^{(-1)} \circ T_\omega \left[\ \eta_c^{\psi_S} + e_{\mathscr{R}}\ V\ \right] \tag{53}$$

Following standard practice we argue heuristically that the term

$$-U^{(-1)} \circ T_\omega\ e_{\mathscr{R}}\ V$$

is negligible because, as we will show in (59), V admits tame estimates. Hence, $U^{(-1)} \circ T_\omega\, e_{\mathscr{R}}\, V$ is quadratic (precise estimates are given in (58)).

We will concentrate in solving a simplified equation, (54), obtained by deleting the quadratically small term from the right-hand side of (53):

$$\Lambda\, V - V \circ T_\omega = -\eta_\Lambda^{\psi_S} \tag{54}$$

where we have denoted

$$\eta_\Lambda^{\psi_S} := U^{(-1)} \circ T_\omega\ \eta_c^{\psi_S}$$

Remark 17. *Note that, as ψ_S is a vector bundle isomorphism, we have that*

$$\eta_\Lambda^{\psi_S} = U^{(-1)} \circ T_\omega\ \psi_S^{-1}\ \eta_c$$

depends linearly on η_c which is an affine function of δ. Hence $\eta_\Lambda^{\psi_S}$ is an affine function of δ. We denote by $\mathtt{Cte}(\eta_\Lambda^{\psi_S})$, $\mathtt{Lin}(\eta_\Lambda^{\psi_S})$ the constant and linear parts of $\eta_\Lambda^{\psi_S}$, respectively. Of course, we have

$$\mathtt{Cte}(\eta_\Lambda^{\psi_S}) = U^{(-1)} \circ T_\omega\ \psi_S^{-1}\ \mathtt{Cte}(\eta_c)\ , \qquad \mathtt{Lin}(\eta_\Lambda^{\psi_S}) = U^{(-1)} \circ T_\omega\ \psi_S^{-1}\ \mathtt{Lin}(\eta_c)$$

If $\left|\ U^{(-1)}\ \right|_\rho\ \le\ C$, $E_S^\sigma \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^\sigma)$, $\sigma \in \{s,u,c\}$, we obtain the estimates:

$$\left|\ \mathtt{Cte}(\eta_\Lambda^{\psi_S})\ \right|_\rho\ \le\ C\ |\ \mathtt{Cte}(\eta_c)\ |_\rho\ , \qquad \left|\ \mathtt{Lin}(\eta_\Lambda^{\psi_S})\ \right|_\rho\ \le\ C\ |\ \mathtt{Lin}(\eta_c)\ |_\rho$$

Lemma 3. *Let $K \in \mathcal{A}_\rho^{\mathbb{C}^c}$, and $\beta \in \mathbb{C}^c$. Let $S^\sigma \in \mathcal{A}_{\rho,0}^{\mathcal{L}\left(E_0^\sigma, E_0^{\sigma'\oplus\sigma''}\right)}$, $\sigma \in \{s,u,c\}$ be such that $E_S^\sigma \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^\sigma)$, $\sigma \in \{s,u,c\}$. Let $U \in \mathcal{A}_\rho^{GL(\mathbb{C}^c\times\mathbb{T}_\rho^d)}$ such that $\left|U^{(-1)}\ \right|_\rho\ \le\ C$. Let $0 < \zeta < \rho$, $\omega \in \mathbb{T}^d$, $\Lambda \in \mathbb{C}^{c\times c}$ such that*

$$\omega \in DC(\nu,\tau),$$

$$Spec(\Lambda) \in DC_\omega^{1st}(\nu,\tau)$$

Let

$$\eta_c \; : \; \mathbb{C}^c \to \mathcal{A}^{E^c_S}_{\rho,\omega}$$
$$\delta \mapsto \eta_c[\delta]. \tag{55}$$

be an affine function of δ.

If the approximate Reducibility identity holds:

$$\overline{A^{c,c}_{K,\beta}}[S] \; U - U \circ T_\omega \; \Lambda = e_{\mathscr{R}} \tag{56}$$

Then, the function

$$\Delta_c \; : \; \mathbb{C}^c \to \mathcal{A}^{E^c_S}_{\rho-\zeta,0}$$
$$\delta \mapsto \Delta_c[\delta].$$

defined by identity (60) *satisfies:*

- Δ_c *is an affine function of* δ.
- *Denote by* $\texttt{Cte}(\Delta_c)$, $\texttt{Lin}(\Delta_c)$ *the constant and linear parts of* Δ_c, *respectively. Then, the following estimates hold:*

$$\begin{aligned} \mid \texttt{Cte}(\Delta_c) \mid_{\rho-\zeta} \; &\leq \; \frac{C}{\zeta^\tau} \;\; \mid \texttt{Cte}(\eta_c) \mid_\rho, \\ \mid \texttt{Lin}(\Delta_\sigma) \mid_{\rho-\zeta} \; &\leq \; \frac{C}{\zeta^\tau} \;\; \mid \texttt{Lin}(\eta_c) \mid_\rho \end{aligned} \tag{57}$$

- Δ_c *is an approximate solution of* (48) *in the following sense: Denote by*

$$e^{\;c}_{qNwt} := \eta_c + A^{c,c}_{K,\beta} \; \Delta_c - \Delta_c \circ T_\omega$$

  the error of the approximate solution. Note that $e^{\;c}_{qNwt}$ *is an affine function of* δ. *Denote by* $\texttt{Cte}(e^{\;c}_{qNwt})$, $\texttt{Lin}(e^{\;c}_{qNwt})$ *the constant and linear parts of* $e^{\;c}_{qNwt}$, *respectively. Then, the following estimates hold:*

$$
\begin{aligned}
\Big|\ \texttt{Cte}(e^{\ c}_{qNwt})\ \Big|_{\rho-\zeta} &\le \frac{C}{\zeta^\tau}\ \ |\ \texttt{Cte}(\eta_c)\ |_\rho\ \ |\ e_{\mathscr{R}}\ |_\rho\ ,\\
\Big|\ \texttt{Lin}(e^{\ c}_{qNwt})\ \Big|_{\rho-\zeta} &\le \frac{C}{\zeta^\tau}\ \ |\ \texttt{Lin}(\eta_c)\ |_\rho\ \ |\ e_{\mathscr{R}}\ |_\rho
\end{aligned}
\tag{58}
$$

Proof. equation (54) is a cohomology equation of the type studied in Subsection 5.5. Since $\Lambda \in DC^{1\text{st}}_\omega(\nu,\tau)$, we can apply Lemma 2: There exists a unique function:

$$
\begin{aligned}
V\ :\ \mathbb{C}^c &\to \mathcal{A}^{\mathbb{C}^c}_{\rho-\zeta}\\
\delta &\mapsto V[\delta]
\end{aligned}
$$

such that:

- V is a solution of (54).
- V is an affine function of δ.
- Denote by $\texttt{Cte}(V)$, $\texttt{Lin}(V)$ the constant and linear parts of V, respectively. From estimates (46), we obtain:

$$
\begin{aligned}
|\ \texttt{Cte}(V)\ |_{\rho-\zeta} &\le \frac{C}{\zeta^\tau}\ \Big|\ \texttt{Cte}(\eta^{\psi_S}_\Lambda)\ \Big|_\rho\\
&\le \frac{C}{\zeta^\tau}\ \ |\ \texttt{Cte}(\eta_c)\ |_\rho\\
|\ \texttt{Lin}(V)\ |_{\rho-\zeta} &\le \frac{C}{\zeta^\tau}\ \Big|\ \texttt{Lin}(\eta^{\psi_S}_\Lambda)\ \Big|_\rho\\
&\le \frac{C}{\zeta^\tau}\ \ |\ \texttt{Lin}(\eta_c)\ |_\rho
\end{aligned}
\tag{59}
$$

Note that V is not a solution of (53). However, it is an approximate solution with error

$$
\begin{aligned}
e^{\psi_S,\ c}_{\mathrm{qNwt}} :=& U^{(-1)}\circ T_\omega\ \big[\ \eta^{\psi_S}_c + e_{\mathscr{R}}\ V\ \big] + \underbrace{\Lambda\ V - V\circ T_\omega}_{-\eta^{\psi_S}_\Lambda}\\
&= U^{(-1)}\circ T_\omega\ e_{\mathscr{R}}\ V + \eta^{\psi_S}_\Lambda - \eta^{\psi_S}_\Lambda\\
&= U^{(-1)}\circ T_\omega\ e_{\mathscr{R}}\ V
\end{aligned}
$$

We will set

$$\Delta_c := \psi_S \; \Delta_c^{\psi_S} = \psi_S \; U \; V \tag{60}$$

as the approximate solution of (48). Finally, note that

$$\begin{aligned} e^c_{\rm qNwt} &= \psi_S \circ T_\omega \; e^{\psi_S,\; c}_{\rm qNwt} \\ &= \big(\; \psi_S \; U^{(-1)} \; \big) \circ T_\omega \; e_{\mathscr{R}} \; V \end{aligned}$$

Remark 18. *Note that Δ_c and e^c_{qNwt} depend linearly on V which is an affine function of δ. Hence Δ_c and e^c_{qNwt} are affine functions of δ. We denote by $\mathtt{Cte}(\Delta_c)$, $\mathtt{Lin}(\Delta_c)$ the constant and linear parts of Δ_c , respectively. Of course, we have*

$$\mathtt{Cte}(\Delta_c) = \psi_S \; U \; \mathtt{Cte}(V) \; , \quad \mathtt{Lin}(\Delta_c) = \psi_S \; U \; \mathtt{Lin}(V)$$

We denote by $\mathtt{Cte}(e^c_{qNwt})$, $\mathtt{Lin}(e^c_{qNwt})$ the constant and linear parts of e^c_{qNwt} , respectively. Of course, we have

$$\begin{aligned} \mathtt{Cte}(e^c_{qNwt}) &= \big(\; \psi_S \; U^{(-1)} \; \big) \circ T_\omega \; e_{\mathscr{R}} \; \mathtt{Cte}(V) \; , \\ \mathtt{Lin}(e^c_{qNwt}) &= \big(\; \psi_S \; U^{(-1)} \; \big) \circ T_\omega \; e_{\mathscr{R}} \; \; \mathtt{Lin}(V) \end{aligned}$$

Since, by assumption, $\big| \; U^{(-1)} \; \big|_\rho \; \leq \; C \; , E^\sigma_S \in \mathcal{V}_{\eta_0^{\mathscr{S}}}\left(E^\sigma_0\right), \; \sigma \in \{s, u, c\}$, we obtain the estimates:

$$\begin{aligned} \mid \mathtt{Cte}(\Delta_c) \mid_{\rho-\zeta} \; &\leq \; \mid \psi_S \; U \mid_{\rho-\zeta} \mid \mathtt{Cte}(V) \mid_{\rho-\zeta} \\ &\leq \; \frac{C}{\zeta^\tau} \; \mid \mathtt{Cte}(\eta_c) \mid_\rho \; , \end{aligned}$$

$$\begin{aligned} \mid \mathtt{Lin}(\Delta_c) \mid_{\rho-\zeta} \; &\leq \; \mid \psi_S \; U \mid_{\rho-\zeta} \mid \mathtt{Lin}(V) \mid_{\rho-\zeta} \\ &\leq \; \frac{C}{\zeta^\tau} \; \mid \mathtt{Lin}(\eta_c) \mid_\rho \end{aligned}$$

These are the estimates (57) in the statement of Lemma 3. Similarly,

$$
\begin{aligned}
\Big| \, \mathtt{Cte}(e^c_{\mathrm{qNwt}}) \, \Big|_{\rho-\zeta} &\leq \Big| \;\; \psi_S \; U^{(-1)} \; \Big|_{\rho-\zeta} \big| \; e_{\mathscr{R}} \; \big|_{\rho-\zeta} \big| \; \mathtt{Cte}(V) \; \big|_{\rho-\zeta} \\
&\leq \; \frac{C}{\zeta^\tau} \;\; \big| \; \mathtt{Cte}(\eta_c) \; \big|_\rho \;\; \big| \; e_{\mathscr{R}} \; \big|_\rho
\end{aligned}
$$

$$
\begin{aligned}
\Big| \, \mathtt{Lin}(e^c_{\mathrm{qNwt}}) \, \Big|_{\rho-\zeta} &\leq \Big| \;\; \psi_S \; U^{(-1)} \; \Big|_{\rho-\zeta} \big| \; e_{\mathscr{R}} \; \big|_{\rho-\zeta} \big| \; \mathtt{Lin}(V) \; \big|_{\rho-\zeta} \\
&\leq \frac{C}{\zeta^\tau} \;\; \big| \; \mathtt{Lin}(\eta_c) \; \big|_\rho \;\; \big| \; e_{\mathscr{R}} \; \big|_\rho
\end{aligned}
$$

These are the estimates (58) in the statement of Lemma 3.

□

Remark 19. *It will be typographically convenient for future applications to introduce the following notation: Let $\widetilde{\Gamma}_{\mathscr{S}^c}$ be the operator acting on affine functions from $\mathbb{C}^c$ taking values in $\mathcal{A}^{E^c_S}_{\rho,0}$ defined by*

$$
\widetilde{\Gamma}_{\mathscr{S}^c} \, (\;\eta_c\;) \, [\delta] := \Delta_c[\delta]
$$

where $\Delta_c[\delta]$ is the unique solution in $\mathcal{A}^{E^c_S}_{\rho,0}$ of equation (48) constructed in Lemma 3.

5.4 Solution of the linearized equations for $\mathscr{S}$.

We now prove Lemma 4, where we construct solutions $\chi^{\sigma,\sigma'}$ of the linear equations for the invariance of the splitting, (61), which are of the form:

$$A_{K,\beta}^{\sigma',\sigma'}\ \chi^{\sigma,\sigma'} - \chi^{\sigma,\sigma'} \circ T_\omega\ \ A_{K,\beta}^{\sigma,\sigma} \ =\ -\ \eta_{\mathscr{S}}^{\sigma,\sigma'}\ , \quad \sigma \neq \sigma' \tag{61}$$

where $\sigma, \sigma' \in \{s, u, c\}$ and $\eta_{\mathscr{S}}^{\sigma,\sigma'}$ is an affine function of δ (as in (62)). The solutions $\chi^{\sigma,\sigma'}$ that we construct in Lemma 4 are affine functions of δ. Estimates for the solution $\chi^{\sigma,\sigma'}$ in terms of the right-hand side of (33), $\eta_{\mathscr{S}}^{\sigma,\sigma'}$, are given in (64) .

In Section 6.2 we will specify the right-hand side term $\eta_{\mathscr{S}}^{\sigma,\sigma'}$ in the linear equation (61) which will be used in the iterative procedure [4].

Lemma 4. *Assume that the splitting E_0 is $\mathtt{N}, C_h, \mu_{K,\beta}^{0,s}, \mu_{K,\beta}^{0,u}, \mu_{K,\beta}^{0,c,+}\ ,\ \mu_{K,\beta}^{0,c,-}$ - whiskered with respect to $A_{K,\beta}$ and, for $\sigma, \sigma' \in \{s, u, c\}$ such that $\sigma \neq \sigma'$, let*

$$\begin{aligned} \eta_{\mathscr{S}}^{\sigma,\sigma'}\ :\ \mathbb{C}^c &\to \mathcal{A}_{\rho,\ \omega}^{\mathcal{L}\left(E_S^{\sigma}, E_S^{\sigma'}\right)} \\ \delta &\mapsto \eta_{\mathscr{S}}^{\sigma,\sigma'}[\delta]. \end{aligned} \tag{62}$$

be affine functions of δ.

Then, there exist unique solutions of (61)

$$\begin{aligned} \chi^{\sigma,\sigma'}\ :\ \mathbb{C}^c &\to \mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_S^{\sigma}, E_S^{\sigma'}\right)} \\ \delta &\mapsto \chi^{\sigma,\sigma'}[\delta]. \end{aligned} \tag{63}$$

defined by equations (66) *and* (68)*, such that:*

- $\chi^{\sigma,\sigma'}$ *are affine functions of δ.*

[4] The statement of Lemma 4 holds for $\eta_{\mathscr{S}}^{\sigma,\sigma'}$ any affine function of δ as in (62).

- *Denote the linear and constant parts of* $\chi^{\sigma,\sigma'}$, *by* $\mathtt{Lin}\left(\chi^{\sigma,\sigma'}\right)$, *and* $\mathtt{Cte}\left(\chi^{\sigma,\sigma'}\right)$, *respectively. Then, the following estimates hold*

$$\begin{aligned} \left| \mathtt{Lin}\left(\chi^{\sigma,\sigma'}\right) \right|_\rho &\le \frac{\mu^{0,c}_{K,\ \beta} \left| \mathtt{Lin}(\eta^{\sigma,\ \sigma'}_{\mathscr{S}}) \right|_\rho}{1-\max\left\{ \mu^{0,s}_{K,\beta}\ \mu^{0,c,-}_{K,\beta},\ \mu^{0,s}_{K,\beta}\ \mu^{0,c,+}_{K,\beta} \right\}} \\ \left| \mathtt{Cte}\left(\chi^{\sigma,\sigma'}\right) \right|_\rho &\le \frac{\mu^{0,c}_{K,\ \beta} \left| \mathtt{Cte}(\eta^{\sigma,\ \sigma'}_{\mathscr{S}}) \right|_\rho}{1-\max\left\{ \mu^{0,s}_{K,\beta}\ \mu^{0,c,-}_{K,\beta},\ \mu^{0,s}_{K,\beta}\ \mu^{0,c,+}_{K,\beta} \right\}} \end{aligned} \tag{64}$$

Proof. Note that a solution $\chi^{\sigma,\sigma'}$ of (61) must satisfy

$$\chi^{\sigma,\sigma'} = -\left(A^{\sigma',\sigma'}_{K,\beta}\right)^{(-1)}\ \eta^{\sigma,\sigma'}_{\mathscr{S}}\ +\ \left(A^{\sigma',\sigma'}_{K,\beta}\right)^{(-1)}\ \chi^{\sigma,\sigma'}\circ T_\omega\ \ A^{\sigma,\sigma}_{K,\beta} \tag{65}$$

Iterating (65) we obtain a candidate expression for $\chi^{\sigma,\sigma'}$:

$$\chi^{\sigma,\sigma'} := \sum_{k=0}^{\infty}\left(\left(A^{\sigma',\sigma'}_{K,\beta}\right)^{(-k-1)}\ \ \left(-\eta^{\sigma,\sigma'}_{\mathscr{S}}\right)\right)\circ T_{k\omega}\ \ \left(A^{\sigma,\sigma}_{K,\beta}\right)^{(k)} \tag{66}$$

The expression in (66) may fail to converge if e.g. $\sigma = u$. To fix this, note that (solving for the second term in the left-hand side of (61)) a solution $\chi^{\sigma,\sigma'}$ of (61) must also satisfy

$$\chi^{\sigma,\sigma'} = \left(\ \eta^{\sigma,\sigma'}_{\mathscr{S}}\left(\ A^{\sigma,\sigma}_{K,\beta}\ \right)^{(-1)}\ \right)\circ T_{-\omega}\ +\ \left(\ A^{\sigma',\sigma'}_{K,\beta}\ \chi^{\sigma,\sigma'}\left(\ A^{\sigma,\sigma}_{K,\beta}\ \right)^{(-1)}\ \right)\circ T_{-\omega} \tag{67}$$

Iterating (67) we obtain another candidate expression for $\chi^{\sigma,\sigma'}$:

$$\chi^{\sigma,\sigma'} := \sum_{k=0}^{\infty}\left(\left(A^{\sigma',\sigma'}_{K,\beta}\right)^{(k)}\ \ \eta^{\sigma,\sigma'}_{\mathscr{S}}\right)\circ T_{-k\omega}\ \ \left(A^{\sigma,\sigma}_{K,\beta}\right)^{(-k-1)} \tag{68}$$

It is easy to verify that both expressions (66), (68) are formal solutions of (61). To obtain convergence of the series, we define $\chi^{\sigma,\sigma'}$ using (66) or (68), according to the rule:

Table 1: Rule for choosing (66) or (68)

If	use
$(\sigma' = s) \vee [\ (\sigma' = c) \wedge (\sigma = u)\ ]$	(68)
else	(66)

Using table 1, we obtain the following candidate expressions:

$$\chi^{c,s} := \sum_{k=0}^{\infty} \left(\left(A^{s,s}_{K,\beta}\right)^{(k)} \quad \eta^{c,s}_{\mathscr{S}} \right) \circ T_{-k\omega} \quad \left(A^{c,c}_{K,\beta}\right)^{(-k-1)}, \quad \chi^{c,u} := \sum_{k=0}^{\infty} \left(\left(A^{u,u}_{K,\beta}\right)^{(-k-1)} \quad (-\eta^{c,u}_{\mathscr{S}}) \right) \circ T_{k\omega} \quad \left(A^{c,c}_{K,\beta}\right)^{(k)}$$

$$\chi^{u,s} := \sum_{k=0}^{\infty} \left(\left(A^{s,s}_{K,\beta}\right)^{(k)} \quad \eta^{u,s}_{\mathscr{S}} \right) \circ T_{-k\omega} \quad \left(A^{u,u}_{K,\beta}\right)^{(-k-1)}, \quad \chi^{u,c} := \sum_{k=0}^{\infty} \left(\left(A^{c,c}_{K,\beta}\right)^{(k)} \quad \eta^{u,c}_{\mathscr{S}} \right) \circ T_{-k\omega} \quad \left(A^{u,u}_{K,\beta}\right)^{(-k-1)}$$

$$\chi^{s,c} := \left(\left(A^{c,c}_{K,\beta}\right)^{(-k-1)} \quad (-\eta^{s,c}_{\mathscr{S}}) \right) \circ T_{k\omega} \quad \left(A^{s,s}_{K,\beta}\right)^{(k)}, \quad \chi^{s,u} := \left(\left(A^{u,u}_{K,\beta}\right)^{(-k-1)} \quad (-\eta^{s,u}_{\mathscr{S}}) \right) \circ T_{k\omega} \quad \left(A^{s,s}_{K,\beta}\right)^{(k)}$$

We now prove the uniform absolute convergence of the series we have used in the candidate expressions:

- if (66) was used, then

$$\begin{aligned}
&\left| \left(\left(A^{\sigma',\sigma'}_{K,\beta}\right)^{(-k-1)} \quad \left(-\eta^{\sigma,\sigma'}_{\mathscr{S}} \right) \right) \circ T_{k\omega} \quad \left(A^{\sigma,\sigma}_{K,\beta}\right)^{(k)} \right|_{\rho} \\
&\leq \left| \left(A^{\sigma',\sigma'}_{K,\beta}\right)^{(-k-1)} \right|_{\rho} \left| \left(A^{\sigma,\sigma}_{K,\beta}\right)^{(k)} \right|_{\rho} \left| \eta^{\sigma,\sigma'}_{\mathscr{S}} \right|_{\rho} \\
&\leq \mu^{0,\sigma'}_{K,\beta} \left(\mu^{0,\sigma}_{K,\beta} \ \ \mu^{0,\sigma'}_{K,\beta} \right)^{k} \left| \eta^{\sigma,\sigma'}_{\mathscr{S}} \right|_{\rho} \\
&\leq \mu^{0,c}_{K,\beta} \left(\max_{j \in s,u} \ \mu^{0,j}_{K,\beta} \ \mu^{0,c}_{K,\beta} \right)^{k} \left| \eta^{\sigma,\sigma'}_{\mathscr{S}} \right|_{\rho}
\end{aligned}$$

  Hence,

$$
\begin{aligned}
\left| \chi^{\sigma,\sigma'} \right|_\rho &= \left| \sum_{k=0}^{\infty} \left(\left(A_{K,\beta}^{\sigma',\sigma'} \right)^{(-k-1)} \quad \left(-\eta_{\mathscr{S}}^{\sigma,\sigma'} \right) \right) \circ T_{k\omega} \quad \left(A_{K,\beta}^{\sigma,\sigma} \right)^{(k)} \right|_\rho \\
&\le \sum_{k=0}^{\infty} \left| \left(\left(A_{K,\beta}^{\sigma',\sigma'} \right)^{(-k-1)} \quad \left(-\eta_{\mathscr{S}}^{\sigma,\sigma'} \right) \right) \circ T_{k\omega} \quad \left(A_{K,\beta}^{\sigma,\sigma} \right)^{(k)} \right|_\rho \\
&\le \sum_{k=0}^{\infty} \mu_{K,\beta}^{0,c} \left(\max_{j \in s,u} \mu_{K,\beta}^{0,j} \quad \mu_{K,\beta}^{0,c} \right)^k \left| \eta_{\mathscr{S}}^{\sigma,\sigma'} \right|_\rho \\
&= \frac{\mu_{K,\ \beta}^{0,c} \left| \eta_{\mathscr{S}}^{\sigma,\ \sigma'} \right|_\rho}{1 - \max_{j \in s,u} \mu_{K,\beta}^{0,j} \quad \mu_{K,\beta}^{0,c}}
\end{aligned}
\tag{69}
$$

(we have used the fact that $0 < \max_{j \in s,u} \mu_{K,\beta}^{0,j} \mu_{K,\beta}^{0,c} < 1$)

- Similarly, if (68) was used,

$$
\left| \left(\left(A_{K,\beta}^{\sigma',\sigma'} \right)^{(k)} \quad \eta_{\mathscr{S}}^{\sigma,\sigma'} \right) \circ T_{-k\omega} \quad \left(A_{K,\beta}^{\sigma,\sigma} \right)^{(-k-1)} \right|_\rho \le \mu_{K,\beta}^{0,c} \left(\max_{j \in s,u} \mu_{K,\beta}^{0,j} \quad \mu_{K,\beta}^{0,c} \right)^k \left| \eta_{\mathscr{S}}^{\sigma,\sigma'} \right|_\rho
$$

Hence,

$$
\begin{aligned}
\left| \chi^{\sigma,\ \sigma'} \right|_\rho &= \left| \sum_{k=0}^{\infty} \left(\left(A_{K,\beta}^{\sigma',\sigma'} \right)^{(k)} \quad \eta_{\mathscr{S}}^{\sigma,\sigma'} \right) \circ T_{-k\omega} \quad \left(A_{K,\beta}^{\sigma,\sigma} \right)^{(-k-1)} \right|_\rho \\
&\le \frac{\mu_{K,\ \beta}^{0,c} \left| \eta_{\mathscr{S}}^{\sigma,\ \sigma'} \right|_\rho}{1 - \max_{j \in s,u} \mu_{K,\beta}^{0,j} \mu_{K,\beta}^{0,c}}
\end{aligned}
\tag{70}
$$

(we have used that $0 < \max_{j \in s,u} \mu_{K,\beta}^{0,j} \quad \mu_{K,\beta}^{0,c} < 1$). Thus, the sums in the candidate expressions converge absolutely in $\mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_S^{\sigma}, E_S^{\sigma'}\right)}$. The absolute convergence justifies the rearrangements needed to show that the series (66), (68) are parameterized solutions of the parameterized linear equations (61).

Remark 20. *Note that* $\chi^{\sigma,\sigma'}$ *depends linearly on* $\eta_{\mathscr{S}}^{\sigma,\sigma'}$*, which is an affine function of* δ*. Thus* $\chi^{\sigma,\sigma'}$ *depends affinely on* δ*. Clearly,*

$$
\texttt{cte}(\chi^{\sigma,\sigma'}) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \Bigg(\left(A_{K,\beta}^{\sigma',\sigma'}\right)^{(k)} \quad \texttt{Cte}(\eta_{\mathscr{S}}^{\sigma,\sigma'}) \Bigg) \circ T_{-k\omega} \quad \left(A_{K,\beta}^{\sigma,\sigma}\right)^{(-k-1)} \\ \qquad\qquad\qquad\qquad\qquad \textit{if } (\sigma' = s) \vee [\ (\sigma' = c) \wedge (\sigma = u)\ ] \\ \displaystyle\sum_{k=0}^{\infty} \Bigg(\left(A_{K,\beta}^{\sigma',\sigma'}\right)^{(-k-1)} \quad \Big(-\texttt{Cte}(\eta_{\mathscr{S}}^{\sigma,\sigma'}) \Big) \Bigg) \circ T_{k\omega} \quad \left(A_{K,\beta}^{\sigma,\sigma}\right)^{(k)} \qquad \textit{else} \end{cases}
$$

$$
\texttt{Lin}(\chi^{\sigma,\sigma'}) = \begin{cases} \displaystyle\sum_{k=0}^{\infty} \Bigg(\left(A_{K,\beta}^{\sigma',\sigma'}\right)^{(k)} \quad \texttt{Lin}(\eta_{\mathscr{S}}^{\sigma,\sigma'}) \Bigg) \circ T_{-k\omega} \quad \left(A_{K,\beta}^{\sigma,\sigma}\right)^{(-k-1)} \\ \qquad\qquad\qquad\qquad\qquad \textit{if } (\sigma' = s) \vee [\ (\sigma' = c) \wedge (\sigma = u)\ ] \\ \displaystyle\sum_{k=0}^{\infty} \Bigg(\left(A_{K,\beta}^{\sigma',\sigma'}\right)^{(-k-1)} \quad \Big(-\texttt{Lin}(\eta_{\mathscr{S}}^{\sigma,\sigma'}) \Big) \Bigg) \circ T_{k\omega} \quad \left(A_{K,\beta}^{\sigma,\sigma}\right)^{(k)} \qquad \textit{else} \end{cases}
$$

Applying estimates (70) to the expressions for $\texttt{Cte}(\chi^{\sigma,\sigma'})$, $\texttt{Lin}(\chi^{\sigma,\sigma'})$ we obtain estimates (64) in the statement. Uniqueness follows easily using an argument analogous to that used for uniqueness in Lemma 1. $\square$

Remark 21. *It will be typographically convenient for future applications to introduce the following notation: For* $\sigma,\ \sigma' \in \{s, u, c\}$*, let* $\widetilde{\Gamma}_{\mathscr{S}^{\sigma,\sigma'}}$ *be the operators acting on affine functions from* $\mathbb{C}^c$ *taking values in* $\mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_S^{\sigma}, E_S^{\sigma'}\right)}$ *defined by*

$$
\widetilde{\Gamma}_{\mathscr{S}^{\sigma,\sigma'}} \Big(\ \eta_{\mathscr{S}}^{\sigma,\sigma'}\ \Big)[\delta] := \chi_{\mathscr{S}}^{\sigma,\sigma'}[\delta]
$$

where $\chi_{\mathscr{S}}^{\sigma,\sigma'}[\delta]$ *is the unique solution of equation (61) in* $\mathcal{A}_{\rho,\ 0}^{\mathcal{L}\left(E_S^{\sigma}, E_S^{\sigma'}\right)}$ *constructed in Lemma 4.*

5.5 Solution of a cohomology equation associated to the linear reducibility equation

In this Section we study the cohomology equation (73). In Lemma 5 we obtain solutions Ω_ξ of (73). Estimates for the solution in terms of the right-hand side of (73) are given in (74). Estimates (74) are standard and well-known in the literature: We reproduce them here for the sake of completeness.

In Subsection 5.5.1 we will see that (73) is a quasi-Newton equation for the improvement for the reducibility functional $\mathscr{R}$. We remark that, in order to solve the cohomology equation (73), the average of each diagonal entry of $\eta_{\mathscr{R},\Lambda}$ must satisfy condition (72). Setting the average in (72) equal to zero leads to choosing the parameter correction γ.

In Section 6.3 we will specify the right-hand side term $\eta_{\mathscr{R},\Lambda}$ in (73) which will be used in the iterative procedure [5].

Lemma 5. *Let* $0 < \zeta < \rho$, $\xi \in \mathbb{C}^c$, $\omega \in \mathbb{T}^d$, $\Lambda = diag[\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_c] \in \mathbb{C}^{c\times c}$, *with*

$$\omega \in DC(\nu,\tau)\ ,$$

$$Spec\,(\Lambda) \in DC^{2nd}_{\omega}(\nu,\tau)$$

Let

$$\begin{aligned} \eta_{\mathscr{R},\Lambda}\ :\ \mathbb{C}^c &\to \mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho,\omega} \\ \delta &\mapsto \eta_{\mathscr{R},\Lambda}[\delta]. \end{aligned} \tag{71}$$

be an affine function of δ.

Assume that there exists $\gamma \in \mathbb{C}^c$ *such that, for any* $1 \le j \le c$,

$$\widehat{\eta_{\mathscr{R},\Lambda}[\gamma]}_{jj}(0) = 0 \tag{72}$$

[5] The statement of Lemma holds for $\eta_{\mathscr{R},\Lambda}$ any affine function of δ as in (71).

(here, $\eta_{\mathscr{R},\Lambda\ jj} \in \mathcal{A}^{\mathbb{C}^c}_{\rho,\omega}$ is the j,j-entry of the matrix $\eta_{\mathscr{R},\Lambda}$).

Then, the function $\Omega_\xi \in \mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho-\zeta,\omega}$ defined by identity (75) is a solution of the cohomology equation:

$$\Lambda\ \Omega_\xi - \Omega_\xi \circ T_\omega\ \Lambda = -\eta_{\mathscr{R},\Lambda}[\gamma] \tag{73}$$

Furthermore:

- *The following estimate holds*

$$|\ \Omega_\xi\ |_{\rho-\zeta}\ \leq\ C\zeta^{-\tau}\,|\ \eta_{\mathscr{R},\Lambda}[\gamma]\ |_\rho \tag{74}$$

- *Ω_ξ is the unique solution of the cohomology equation (73) in $\mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho-\zeta,\omega}$ which satisfies:*

$$\widehat{\Omega_{\xi,jj}}(0) = \xi_j\ , \quad \textit{for}\ \ 1 \leq j \leq c$$

Proof. Let $\Omega_{\xi,jl} \in \mathcal{A}^{\mathbb{C}^c}_{\rho-\zeta,\omega}$ be the jl-entry of the matrix Ω_ξ and let $\eta_{\mathscr{R},\Lambda\ jl}[\gamma] \in \mathcal{A}^{\mathbb{C}^c}_{\rho,\omega}$ be the jl-entry of the matrix $\eta_{\mathscr{R},\Lambda}[\gamma]$. Using the Fourier method we see that (73) holds if, and only if,

$$\widehat{\Omega_{\xi,jl}}(k)\,(\ \lambda_l - \lambda_j\ \ \exp(2\pi i k\cdot\omega)) = -\widehat{\eta_{\mathscr{R},\Lambda}}_{jl}[\gamma](k)$$

for all $k \in \mathbb{Z}^d$, $1 \leq j,l \leq c$. Since $\Lambda \in DC^{2nd}_\omega(\nu,\tau)$, the factor

$$\lambda_l - \lambda_j\ \ \exp(2\pi i k\cdot\omega)$$

is not zero whenever $k \neq 0$ or $j \neq l$ and we can define Ω_ξ formally as having Fourier coefficients

$$\widehat{\Omega_\xi}_{jl}(k) := \begin{cases} -\dfrac{\widehat{\eta^{\psi_S}_{\Lambda,j}}[\gamma](k)}{\lambda_l - \lambda_j \exp(2\pi i k\cdot\omega)} & \text{if}\ \ k \neq 0,\ \ \text{or}\ \ l \neq j \\ \\ \xi_j & \text{otherwise} \end{cases} \tag{75}$$

The estimates (74) for the twisted cohomology equations (73) have been obtained by Eliasson [13] using also techniques developed by Rüssmann [57] for non-twisted equations.

The expression (75) for the Fourier coefficients of Ω_ξ defines Ω_ξ uniquely in $\mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho-\zeta,\omega}$

$\square$

5.5.1 Approximate solution of the linear equation corresponding to the Reducibility.

We now prove Lemma 6: Given any $\xi \in \mathbb{C}^c$, we construct *approximate* solutions $(W_\xi,\ \gamma)$ of the linear Reducibility equation in the central directions, which is of the form:

$$\overline{A^{c,c}_{K,\beta}}[S]\ W_\xi - W_\xi \circ T_\omega\ \Lambda = -\eta_{\mathscr{R}}[\gamma] \tag{76}$$

where $\eta_{\mathscr{R}}$ is an affine function (as in (83)).

Estimates for the approximate solution $(W_\xi,\ \gamma)$ in terms of the right-hand side of (76), $\eta_{\mathscr{R}}$, are given in (57).

We remark that there are two unknowns of equation (76): The function W_ξ and the parameter γ. The parameter γ is chosen so that the cohomology equation (80) is solvable. In Lemma 6 we will find W_ξ and γ such that equation (76) is satisfied approximately, with error $e^{\mathscr{R}}_{\mathrm{qNwt}}$ which will be estimated in (88).

In Section 6.3 we will specify the affine function $\eta_{\mathscr{R}}$, in the right-hand side of equation (76), which will be used in the iterative procedure [6].

We begin by using the Approximate Reducibility identity

$$\overline{A^{c,c}_{K,\beta}}[S]\ U - U \circ T_\omega\ \Lambda = e_{\mathscr{R}} \tag{77}$$

to approximately reduce equation (76) into a constant coefficients equation, (80) which is furthermore diagonalized.

[6]Since η_c is not yet specified, the statement of Lemma 3 holds for η_c any affine function of δ as in (83).

5.5.1.1 Approximate reducibility to constant coefficients: Cohomology equation for Reducibility.

By (77) we have

$$\overline{A^{c,c}_{K,\beta}}[S] = (\ e_{\mathscr{R}}\ +\ U \circ T_\omega\ \Lambda\ )\ U^{(-1)}$$

Substituting this expression for $\overline{A^{c,c}_{K,\beta}}[S]$ in (76) we obtain an equivalent equation:

$$(\ e_{\mathscr{R}}\ +\ U \circ T_\omega\ \Lambda\ )\ U^{(-1)}\ W_\xi - W_\xi \circ T_\omega\ \Lambda = -\ \eta_{\mathscr{R}} \tag{78}$$

We will not solve (78), but rather the simplified equation obtained as before by deleting the quadratic term

$$e_{\mathscr{R}}\ U^{(-1)}\ W_\xi$$

from the left hand side of (78):

$$\Lambda\ U^{(-1)}\ W_\xi - \left(\ U^{(-1)}\ W_\xi\ \right) \circ T_\omega\ \Lambda = -\ U^{(-1)} \circ T_\omega\ \eta_{\mathscr{R}} \tag{79}$$

with the change of variables

$$W_\xi = U\ \Omega_\xi$$

rewrite (79) as

$$\Lambda\ \Omega_\xi - \Omega_\xi \circ T_\omega\ \Lambda = -\eta_{\mathscr{R},\ \Lambda} \tag{80}$$

where we have denoted

$$\eta_{\mathscr{R},\ \Lambda} :=\ U^{(-1)} \circ T_\omega\ \eta_{\mathscr{R}}$$

Remark 22. *Note that $\eta_{\mathscr{R},\ \Lambda}$ is a linear function of $\eta_{\mathscr{R}}$, which is an affine function of δ. Thus, $\eta_{\mathscr{R},\ \Lambda} = \eta_{\mathscr{R},\ \Lambda}[\delta]$ depends affinely on δ. Of course we have:*

$$\mathtt{Lin}(\eta_{\mathscr{R},\ \Lambda})\ =\ U^{(-1)} \circ T_\omega\ \mathtt{Lin}(\eta_{\mathscr{R}})\ ,\quad \mathtt{Cte}(\eta_{\mathscr{R},\ \Lambda})\ =\ U^{(-1)} \circ T_\omega\ \mathtt{Cte}(\eta_{\mathscr{R}})$$

If $\left|\ U^{(-1)}\ \right|_\rho\ \leq\ C$, we obtain the estimates

$$\begin{aligned} \left|\ \mathtt{Lin}(\eta_{\mathscr{R},\Lambda})\ \right|_\rho\ &\leq C\ \left|\ \mathtt{Lin}(\eta_{\mathscr{R}})\ \right|_\rho \\ \left|\ \mathtt{Cte}(\eta_{\mathscr{R},\Lambda})\ \right|_\rho\ &\leq C\ \left|\ \mathtt{Cte}(\eta_{\mathscr{R}})\ \right|_\rho \end{aligned} \tag{81}$$

A solution of the simplified equation (79) is not a solution of (76). However, it is an *approximate* solution with error

$$\begin{aligned} e^{\mathscr{R}}_{\mathrm{qNwt}} &:= e_{\mathscr{R}}\, U^{(-1)}\, W_\xi \;+\; \underbrace{U\circ T_\omega\ \Lambda\, U^{(-1)}\, W_\xi - W_\xi\circ T_\omega\ \Lambda}_{=\,-\eta_{\mathscr{R}}} + \eta_{\mathscr{R}} \\ &= e_{\mathscr{R}}\, U^{(-1)}\, W_\xi \end{aligned} \tag{82}$$

Lemma 6. *Let $0<\zeta<\rho$, $\xi\in\mathbb{C}^c$, $\omega\in\mathbb{T}^d$, $\Lambda\in\mathbb{C}^{c\times c}$ be as in Lemma 5. Let*

$$\begin{aligned} \eta_{\mathscr{R}} \;:\; \mathbb{C}^c &\to \mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho,\omega} \\ \delta &\mapsto \eta_{\mathscr{R}}[\delta]. \end{aligned} \tag{83}$$

an affine function of δ. Let $U\in\mathcal{A}_\rho^{GL(\mathbb{C}^c\times\mathbb{T}^d_\rho)}$ be such that $\left|\, U^{(-1)}\,\right|_\rho \le C$. Let $S^\sigma\in\mathcal{A}_{\rho,0}^{\mathcal{L}(E_0^\sigma, E_0^{\sigma'}\oplus\sigma'')}$, $\sigma\in\{s,u,c\}$, where $\{\sigma,\sigma',\sigma''\}$ is a permutation of the symbols $\{s,u,c\}$, and assume that $E_S^\sigma\in\mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^\sigma)$.

Denote by $e_{\mathscr{R}}$ the error of the approximate reducibility equation:

$$\overline{A^{c,c}_{K,\beta}}[\psi_S]\; U - U\circ T_\omega\ \Lambda = e_{\mathscr{R}} \tag{84}$$

Assume also that the following non-degeneracy condition holds:

$$M := \left(\min_{1\le j\le c}\left|\ [\ \mathtt{Lin}(\eta_{\mathscr{R},\Lambda})\ ]^{\widehat{}}_{jj}\,(0)\ \right|\right)^{-1} < \infty \tag{85}$$

Then, the function $W_\xi\in\mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho-\zeta,0}$ defined by identity (93), and the correction for the parameter $\gamma\in\mathbb{C}^c$ defined by identity (91) satisfy:

- *The following estimates hold:*

$$|\ W_\xi\ |_{\rho-\zeta}\ \le\ \frac{C\ M}{\zeta^\tau}\ \ |\ \mathtt{Cte}(\ \eta_{\mathscr{R}}\ )\ |_\rho\ \ |\ \mathtt{Lin}(\ \eta_{\mathscr{R}}\ )\ |_\rho \tag{86}$$

$$|\ \gamma\ |\ \le\ C\ M\ \ |\ \mathtt{Cte}(\eta_{\mathscr{R}})\ |_\rho \tag{87}$$

- $(W_\xi,\ \gamma\ )$ *is an approximate solution of* (48) *in the following sense: Denote by*

$$e^{\mathscr{R}}_{qNwt} := \overline{A^{c,c}_{K,\beta}}[S]\ W_\xi - W_\xi \circ T_\omega + \eta_{\mathscr{R}}[\gamma]$$

the error of the approximate solution. Then, the following estimate holds:

$$\left|\ e^{\mathscr{R}}_{qNwt}\ \right|_{\rho-\zeta}\ \leq\ \frac{C\ M}{\zeta^\tau}\ \left|\ e_{\mathscr{R}}\ \right|_\rho\ \left|\ \mathtt{Cte}(\ \eta_{\mathscr{R}}\ )\ \right|_\rho\ \left|\ \mathtt{Lin}(\ \eta_{\mathscr{R}}\ )\ \right|_\rho \tag{88}$$

Proof. Equation (80) is a cohomology equation of the type studied in Subsection 5.5. To apply Lemma 5, we must check the condition:

$$[\ \eta_{\mathscr{R},\ \Lambda}[\gamma]\ ]\widehat{\ }_{jj}\,(0) = 0\ ,\quad 1 \leq j \leq c \tag{89}$$

Recall from Remark 22 that $\eta_{\mathscr{R},\ \Lambda}$ is an affine function. Hence, condition (89) holds if, and only if,

$$[\ \mathtt{Lin}(\eta_{\mathscr{R},\ \Lambda})\ \gamma\ ]\widehat{\ }_{jj}\,(0) = -\,[\ \mathtt{Cte}(\eta_{\mathscr{R},\ \Lambda})\ ]\widehat{\ }_{jj}\,(0)$$

that is,

$$[\ \mathtt{Lin}(\eta_{\mathscr{R},\ \Lambda})\ ]\widehat{\ }_{jj}\,(0)\ \gamma_j = -\ [\ \mathtt{Cte}(\ \eta_{\mathscr{R},\ \Lambda}\ )\ ]\widehat{\ }_{jj}\,(0) \tag{90}$$

for $1 \leq j \leq c$, where $\gamma = (\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_c)^{\mathtt{t}}$.

Note that the Non-degeneracy condition (85) implies that there is a unique solution $\gamma \in \mathbb{C}^c$ of the linear equations (90)

$$\gamma_j = -\,\frac{[\ \mathtt{Cte}(\ \eta_{\mathscr{R},\ \Lambda}\ )\ ]\widehat{\ }_{jj}\,(0)}{[\ \mathtt{Lin}(\eta_{\mathscr{R},\ \Lambda})\ ]\widehat{\ }_{jj}\,(0)} \tag{91}$$

Using the Cauchy-Fourier estimate, (81) and (85), we obtain

$$\begin{aligned} \left|\ \gamma\ \right|\ &\leq\ M\ \left|\ \mathtt{Cte}(\ \eta_{\mathscr{R},\ \Lambda}\ )\ \right|_\rho \\ &\leq\ C\ M\ \left|\ \mathtt{Cte}(\eta_{\mathscr{R}})\ \right|_\rho \end{aligned}$$

Hence, we fix $\eta_{\mathscr{R},\ \Lambda}\ [\gamma]$ so condition (85) holds and we can apply Lemma 5 to obtain a unique solution $\Omega_\xi \in \mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho-\zeta}$ of (80) such that $\widehat{\Omega_\xi}(0) \ = \ \xi$. We also have the estimate

$$\begin{aligned} |\ \Omega_\xi\ |_{\rho-\zeta} \ &\leq \frac{1}{\zeta^\tau} |\ \ \eta_{\mathscr{R},\ \Lambda}[\gamma]\ |_\rho \\ &\leq \frac{1}{\zeta^\tau}\ \Big[\ |\ \mathtt{Cte}(\ \eta_{\mathscr{R},\ \Lambda}\ )\ |_\rho + |\ \mathtt{Lin}(\ \eta_{\mathscr{R},\ \Lambda}\ )\ |_\rho\ \ |\ \gamma\ |\ \Big] \\ &\leq \frac{C\ M}{\zeta^\tau}\ \ |\ \mathtt{Cte}(\ \eta_{\mathscr{R}}\ )\ |_\rho\ \ |\ \mathtt{Lin}(\ \eta_{\mathscr{R}}\ )\ |_\rho \end{aligned} \tag{92}$$

Recall that we had set

$$W_\xi = U\ \Omega_\xi \tag{93}$$

Using estimates (92), we obtain estimate (86) in the statement.

We had found in (82) that $(\ W_\xi,\ \gamma\ )$ is an approximate solution of (76) with error

$$e^{\mathscr{R}}_{\mathrm{qNwt}} = e_{\mathscr{R}}\ U^{(-1)}\ W_\xi$$

Hence, using estimate (86), estimate (88) in the statement follows.

$\square$

Remark 23. *It will be typographically convenient for future applications to introduce the following notation: Given $\xi \in \mathbb{C}^c$, let $\left(\ \widetilde{\Gamma}_{\mathscr{R},\xi},\ \tilde{\gamma}\ \right)$ be the operator acting on affine functions, and taking values in $\mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho-\zeta} \times \mathbb{C}^c$, defined by*

$$\left(\ \widetilde{\Gamma}_{\mathscr{R},\xi}\ (\ \eta_{\mathscr{R}}\ )\ ,\ \ \tilde{\gamma}(\eta_{\mathscr{R}})\ \right) := (\ W_\xi,\ \ \gamma\ )$$

where $(\ W_\xi,\ \gamma\ )$ is the unique approximate solution of equation (76) in $\mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho-\zeta,\ 0}$ constructed in Lemma 6.

CHAPTER VI

ZEHNDER'S APPROXIMATE RIGHT INVERSES FOR $D\mathscr{I}$, $D\mathscr{S}$, $D\mathscr{R}$.

In this Chapter we use the operators $\widetilde{\Gamma}_{\mathscr{I}^\sigma}$, $\sigma \in \{s,u,c\}$, $\widetilde{\Gamma}_{\mathscr{S}^{\sigma,\sigma'}}$, $\sigma \neq \sigma'$, $\widetilde{\Gamma}_{\mathscr{R},\xi}$, $\tilde{\gamma}$ defined in Section 5 in order to construct approximate right inverses for $D\mathscr{I}$, $D\mathscr{S}$ and $D\mathscr{R}$, as outlined in Section 4.3. The approximate right inverses will be used in the iterative procedure outlined in Algorithm 1.

6.1 *Approximate right inverse for* $D\mathscr{I}$

Finding an approximate right inverse for $D\mathscr{I}$ is equivalent to solving the Newton equation

$$D\mathscr{I}\,[K,\ \beta;\ \Gamma_{\mathscr{I}}(\ \eta\ )[\delta],\ \delta] + \eta\ \equiv\ 0, \quad \delta \in \mathbb{C}^c,\ \eta \in \mathcal{A}_\rho^{\mathbb{C}^n} \tag{94}$$

In Lemma 7, using the splitting E_S and the expression in equation (97) for $D\mathscr{I}$, we first see that the Newton equation (94) is equivalent to a system of three linear equations. We use the operators $\widetilde{\Gamma}_{\mathscr{I}^\sigma}$ to approximately solve the system (see equations (98), (99)). Adding the approximate solutions of the system we obtain an approximate solution of (94) (see (100)), and we give an estimate for the error of the approximate solution in (95).

Lemma 7. *Let* $0 < \zeta < \rho$, $K \in \mathcal{A}_\rho^{\mathbb{C}^n}$, $\beta \in \mathbb{C}^c$, $S^\sigma \in \mathcal{A}_{\rho,0}^{\mathcal{L}(E_0^\sigma, E_0^{\sigma'\oplus\sigma''})}$, $\sigma \in \{s,u,c\}$, *where* $\{\sigma,\sigma',\sigma''\}$ *is a permutation of the symbols* $\{s,u,c\}$. *Assume that the splitting* E_S *is* $\mathtt{N}$, C_h, $\mu_{K,\beta}^{S,s}$, $\mu_{K,\beta}^{S,u}$, $\mu_{K,\beta}^{S,c,+}$, $\mu_{K,\beta}^{S,c,-}$*-whiskered with respect to* $A_{K,\beta}$, *and assume that* $E_S^\sigma \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^\sigma)$. *Let* $U \in \mathcal{A}_\rho^{GL(c,\mathbb{C})}$ *be such that* $\left|\ U^{(-1)}\ \right|_\rho \leq C$. *Let* $\omega \in \mathbb{T}^d$,

$\Lambda \in \mathbb{C}^{c\times c}$, $\Lambda = diag[\lambda_1, \ \ldots, \ \lambda_c]$, *be such that*

$$\omega \in DC(\nu, \tau),$$

$$Spec(\Lambda) \in DC^{1st}_{\omega}(\nu, \tau)$$

Assume that $K(\mathbb{T}^d_\rho) \subset \mathcal{D}_{\rho_0,\eta_0}(K_0, \beta_0)$. *We denote the error of the invariance equation for the torus by*

$$e := \mathscr{I}[K, \beta].$$

Recall from Remarks 16, 21 the operators acting on affine functions $\widetilde{\Gamma}_{\mathscr{I}^\sigma}$, $\sigma \in \{s, u, c\}$. *Define an operator* $\Gamma_{\mathscr{I}}$ *acting on* $\mathcal{A}^{\mathbb{C}^n}_\rho$ *and taking values on affine functions as follows: Given* $\eta \in \mathcal{A}^{\mathbb{C}^n}_\rho$, *let*

$$\Gamma_{\mathscr{I}}(\eta) \ := \ \sum_{\sigma\in\{s,u,c\}} \widetilde{\Gamma}_{\mathscr{I}^\sigma} \left(\ \Pi^\sigma_S \circ T_\omega \ \left(\ \eta \ + \ \partial_\beta f_\beta \circ K \ \delta \ \right) \ \right)$$

We will denote $\tilde{\eta}_\sigma := \Pi^\sigma_S \circ T_\omega \ (\ \eta \ + \ \partial_\beta f_\beta \circ K \ \delta \ )$, *so that*

$$\Gamma_{\mathscr{I}}(\eta) \ = \ \sum_{\sigma\in\{s,u,c\}} \widetilde{\Gamma}_{\mathscr{I}^\sigma} \left(\ \tilde{\eta}_\sigma \ \right)$$

Denote by $e_{\mathscr{R}}$ *the error of the approximate reducibility equation:*

$$\overline{A^{c,c}_{K,\beta}}[S] \ U - U \circ T_\omega \ \Lambda = e_{\mathscr{R}}$$

Denote by $e_{\mathscr{S}}$ *the error of the approximate invariance of the splitting equation:*

$$\mathscr{S}^\sigma[K, \ \beta, \ S^{\sigma,\sigma'}, \ S^{\sigma,\sigma''}] \ := \ e^\sigma_{\mathscr{S}}$$

and let $e_{\mathscr{S}} := (e^s_{\mathscr{S}}, \ e^u_{\mathscr{S}}, \ e^c_{\mathscr{S}})$.

Then, given any $\delta \in \mathbb{C}^c$, *the following estimate holds:*

$$
\begin{aligned}
&| \ D\mathscr{I} \, [\ K, \ \beta; \ \Gamma_{\mathscr{I}}(\ \eta \ )[\delta], \ \delta \ ] \ + \ \eta \ |_{\rho-\zeta} \\
&\quad \leq \ C \ \ (\ | \ \eta \ |_{\rho} \ + \ | \ \partial_{\beta} \ f_{\beta} \circ K \ |_{\rho} \ | \ \delta \ | \ ) \, \frac{| \ e_{\mathscr{S}} \ |_{\rho} \ + \ | \ e_{\mathscr{R}} \ |_{\rho}}{\zeta^{\tau}}
\end{aligned} \tag{95}
$$

Proof. Given any $\delta \in \mathbb{C}^c$, consider the linearization

$$
\begin{aligned}
\mathscr{I}[K + \Gamma_{\mathscr{I}}(\ \eta \ )[\delta], \ \beta + \delta] \ &= e + A_{K,\beta} \, \Gamma_{\mathscr{I}}(\ \eta \ )[\delta] - \Gamma_{\mathscr{I}}(\ \eta \ )[\delta] \circ T_{\omega} \\
&+ \partial_{\beta} f_{\beta} \circ K \ \delta + \mathcal{E}_2(\mathscr{I})[K, \ \beta; \ \Gamma_{\mathscr{I}}(\ \eta \ )[\delta], \ \delta]
\end{aligned} \tag{96}
$$

From (96) we see that

$$
\begin{aligned}
D\mathscr{I} \, [K, \ \beta; \ \Gamma_{\mathscr{I}}(\ \eta \ )[\delta], \ \delta] \ &= A_{K,\beta} \, \Gamma_{\mathscr{I}}(\ \eta \ )[\delta] \ - \ \Gamma_{\mathscr{I}}(\ \eta \ )[\delta] \circ T_{\omega} \\
&+ \partial_{\beta} f_{\beta} \circ K \ \delta
\end{aligned} \tag{97}
$$

Using Lemma 1, we have:

$$
\begin{cases}
A^{s,s}_{K,\beta}[S] \ \widetilde{\Gamma}_{\mathscr{I}^s}(\ \tilde{\eta}_s \ ) - \widetilde{\Gamma}_{\mathscr{I}^s}(\ \tilde{\eta}_s \ ) \circ T_{\omega} = -\tilde{\eta}_s \\
A^{u,u}_{K,\beta}[S] \ \widetilde{\Gamma}_{\mathscr{I}^u}(\ \tilde{\eta}_u \ ) - \widetilde{\Gamma}_{\mathscr{I}^u}(\ \tilde{\eta}_u \ ) \circ T_{\omega} = -\tilde{\eta}_u
\end{cases} \tag{98}
$$

Using Lemma 3, we have:

$$
A^{c,c}_{K,\beta}[S] \ \widetilde{\Gamma}_{\mathscr{I}^c}(\ \tilde{\eta}_c \ ) - \widetilde{\Gamma}_{\mathscr{I}^c}(\ \tilde{\eta}_c \ ) \circ T_{\omega} = -\tilde{\eta}_c \ + e^{c}_{qNwt} \tag{99}
$$

Adding the three identities in (98) and (99) we obtain

$$
\begin{aligned}
&A_{K,\beta} \, \Gamma_{\mathscr{I}}(\ \eta \ )[\delta] \ - \ \Gamma_{\mathscr{I}}(\ \eta \ )[\delta] \circ T_{\omega} \\
&\qquad = \ - \eta - \partial_{\beta} f_{\beta} \circ K \ \delta \ \ + \ e^{c}_{qNwt}[\delta] \ + \ e_{\Pi_S}[\delta]
\end{aligned}
$$

where we have denoted by e_{Π_S} the out-of-diagonal terms

$$e_{\Pi_S} \ := \ \sum_{\sigma \neq \sigma'} A^{\sigma,\sigma'}_{K,\beta}[S] \ \widetilde{\Gamma}_{\mathscr{I}^\sigma}(\ \tilde{\eta}_\sigma \ )$$

Hence, using (97) it holds that

$$D\mathscr{I}\,[K, \ \beta; \ \Gamma_{\mathscr{I}}(\ \eta \ )[\delta], \ \delta] + \eta \ = \ e^c_{qNwt}[\delta] \ + \ e_{\Pi_S}[\delta] \tag{100}$$

In Lemma 3 we had obtained estimates (58) for e^c_{qNwt}, which give:

$$\begin{aligned} \Big| \ e^{\,c}_{\mathrm{qNwt}} \ \Big|_{\rho-\zeta} \ &\leq \ \Big| \ \mathtt{Cte}(e^{\,c}_{\mathrm{qNwt}}) \ \Big|_{\rho-\zeta} \ + \ \Big| \ \mathtt{Lin}(e^{\,c}_{\mathrm{qNwt}}) \ \Big|_{\rho-\zeta} \ | \ \delta \ | \\ &\leq (\ | \ \mathtt{Cte}(\tilde{\eta}_c) \ |_\rho \ + \ | \ \mathtt{Lin}(\tilde{\eta}_c) \ |_\rho \ | \ \delta \ | \ ) \ \frac{C}{\zeta^\tau} \ | \ e_{\mathscr{R}} \ |_\rho \end{aligned} \tag{101}$$

Note that

$$\mathtt{Cte}(\tilde{\eta}_c) = \Pi^c_S \circ T_\omega \ \eta \ , \qquad \mathtt{Lin}(\tilde{\eta}_c) = \Pi^c_S \circ T_\omega \ \partial_\beta \ f_\beta \circ K$$

Hence,

$$\begin{aligned} \Big| \ e^{\,c}_{\mathrm{qNwt}} \ \Big|_{\rho-\zeta} \ &\leq \ (\ | \ \eta \ |_\rho \ + \ | \ \partial_\beta \ f_\beta \circ K \ |_\rho \ | \ \delta \ | \ ) \ \frac{C}{\zeta^\tau} \ | \ e_{\mathscr{R}} \ |_\rho \\ &\leq \frac{C}{\zeta^\tau} \ | \ e_{\mathscr{R}} \ |_\rho \, (\ | \ \eta \ |_\rho \ + \ | \ \delta \ | \ ) \end{aligned} \tag{102}$$

Recall from Proposition 1 that

$$\Big| \ A^{\sigma,\sigma'}_{K,\beta} \ [S] \ \Big|_\rho \leq C \, | \ A_{K,\beta} \ |_\rho \ | \ e_{\mathscr{S}} \ |_\rho$$

Hence,

$$\begin{aligned} | \ e_{\Pi_S}[\delta] \ |_{\rho-\zeta} \ &\leq \ \sum_{\sigma \neq \sigma'} \Big| \ A^{\sigma,\sigma'}_{K,\beta} \ [S] \ \Big|_\rho \ \Big| \ \widetilde{\Gamma}_{\mathscr{I}^\sigma}(\ \tilde{\eta}_\sigma \ )[\delta] \ \Big|_\rho \\ &\leq \ (\ | \ \eta \ |_\rho \ + \ | \ \delta \ | \ ) \ \frac{C}{\zeta^\tau} \ | \ e_{\mathscr{S}} \ |_\rho \end{aligned} \tag{103}$$

Adding estimates (102) and (103), and using identity (100) we obtain estimate (95) in the statement of Lemma 7. $\square$

6.2 Right inverse for $D\mathscr{S}$

Finding a right inverse $\Gamma_{\mathscr{S}^\sigma} := (\Gamma_{\mathscr{S}^{\sigma,\sigma'}},\ \Gamma_{\mathscr{S}^{\sigma,\sigma''}})$ for $D\mathscr{S}^\sigma$ is equivalent to solving the Newton equation

$$
\begin{aligned}
&D\mathscr{S}^\sigma\Big[\ K,\ \beta,\ S^{\sigma,\sigma'},\ S^{\sigma,\sigma''};\\
&\qquad \Delta[\delta],\ \delta,\ \Gamma_{\mathscr{S}^{\sigma,\sigma'}}(\eta^{\sigma,\sigma'};\ \Delta)[\delta],\ \Gamma_{\mathscr{S}^{\sigma,\sigma''}}(\eta^{\sigma,\sigma''};\ \Delta)[\delta]\ \Big]\ +\ \eta^\sigma[\delta] \equiv\ 0
\end{aligned}
$$

for any $(\eta^{\sigma,\sigma'}, \eta^{\sigma,\sigma''})$ affine functions as in (106), $\eta := \left(\ \eta^{\sigma,\sigma'},\ \eta^{\sigma,\sigma''}\ \right)^{\mathtt{t}}$, Δ an affine function[1] as in (104), and $\delta \in \mathbb{C}^c$. In Lemma 8 we first obtain an expression for

$$
D\mathscr{S}^\sigma\Big[\ K, \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''};\ \Delta[\delta], \delta, \Gamma_{\mathscr{S}^{\sigma,\sigma'}}(\eta^{\sigma,\sigma'};\Delta)[\delta],\ \Gamma_{\mathscr{S}^{\sigma,\sigma''}}(\eta^{\sigma,\sigma''};\Delta)[\delta]\ \Big]
$$

which requires to introduce the affine function $\eta^{\sigma,\sigma'}_{\mathscr{S},\Delta}$, defined in equation (105). Then we use the operators $\widetilde{\Gamma}_{\mathscr{S}^{\sigma,\sigma'}}$ introduced in Remark 21 to solve exactly the Newton equation.

Lemma 8. *Let $(\sigma,\ \sigma',\ \sigma'')$ be a permutation of the letters (s, c, u). Let $0 < \rho$, $K \in \mathcal{A}^{\mathbb{C}^n}_\rho$, $\beta \in \mathbb{C}^c$, and $S^\sigma \in \mathcal{A}^{\mathcal{L}(E^\sigma_0, E^{\sigma'\oplus\sigma''}_0)}_{\rho,0}$ for $\sigma \in \{s, c, u\}$. Assume that $E^\sigma_S \in \mathcal{V}_{\eta^{\mathscr{S}}_0}(E^\sigma_0)$, and assume that the splitting E_0 is $\mathtt{N}$, C_h, $\mu^{0,s}_{K,\beta}$, $\mu^{0,u}_{K,\beta}$, $\mu^{0,c,+}_{K,\beta}$, $\mu^{0,c,-}_{K,\beta}$-whiskered with respect to $A_{K,\beta}$.*

Denote the error of the invariance equation for E^σ_S as

$$
\mathscr{S}^\sigma[K,\ \beta,\ S^{\sigma,\sigma'},\ S^{\sigma,\sigma''}]\ :=\ e^\sigma_{\mathscr{S}}\ :=\ \left(e^{\sigma,\sigma'}_{\mathscr{S}}, e^{\sigma,\sigma''}_{\mathscr{S}}\right)^{\mathtt{t}}
$$

Let

$$
\begin{aligned}
\Delta\ :\ \mathbb{C}^c &\to \mathcal{A}^{\mathbb{C}^n}_{\rho,0}\\
\delta &\mapsto \Delta[\delta].
\end{aligned}
\tag{104}
$$

[1] Note that we use the affine function Δ as a parameter for the operator $\Gamma_{\mathscr{S}^\sigma}$

be an affine function.

Given $\sigma \neq \sigma'$, *let* $\eta^{\sigma,\sigma'}_{\mathscr{S},\Delta}$ *be the affine function of* $\delta \in \mathbb{C}^c$ *and taking values in* $\mathcal{A}^{\mathcal{L}(E^{\sigma}_0, E^{\sigma'}_0)}_{\rho,\omega}$ *defined by*

$$\begin{aligned} \eta^{\sigma,\sigma'}_{\mathscr{S},\Delta}[\delta] \ :=\ & \Big(\ DA^{\sigma',\sigma'}_{K,\beta}\ \Delta[\delta]\ +\ \partial_\beta A^{\sigma',\sigma'}_{K,\beta}\ \delta\ \Big)\, S^{\sigma,\sigma'} \\ & -\ S^{\sigma,\sigma'} \circ T_\omega\,\big(\ DA^{\sigma,\sigma}_{K,\beta}\ \Delta[\delta]\ +\ \partial_\beta A^{\sigma,\sigma}_{K,\beta}\ \delta\ \big) \end{aligned} \tag{105}$$

and let

$$\eta^{\sigma}_{\mathscr{S},\Delta} := \Big(\ \eta^{\sigma,\sigma'}_{\mathscr{S},\Delta},\ \eta^{\sigma,\sigma''}_{\mathscr{S},\Delta}\ \Big)^{\mathtt{t}}$$

Recall from Remark 21 the operators acting on affine functions $\widetilde{\Gamma}_{\mathscr{S}^{\sigma,\sigma'}}$. *Define an operator acting on affine functions and taking values on affine functions as follows: Given*

$$\begin{aligned} \eta^{\sigma,\sigma'}\ :\ \mathbb{C}^c &\to \mathcal{A}^{\mathcal{L}(E^{\sigma}_0, E^{\sigma'}_0)}_{\rho,0} \\ \delta &\mapsto \eta^{\sigma,\sigma'}[\delta]. \end{aligned} \tag{106}$$

an affine function, let

$$\Gamma_{\mathscr{S}^{\sigma,\sigma'}}(\ \eta^{\sigma,\sigma'};\ \Delta\ ) := \ \widetilde{\Gamma}_{\mathscr{S}^{\sigma,\sigma'}}(\ \eta^{\sigma,\sigma'}\ +\ \eta^{\sigma,\sigma'}_{\mathscr{S},\Delta}\ )$$

When typographically convenient, we will denote

$$\Gamma_{\mathscr{S}^{\sigma,\sigma'}}(\ \eta^{\sigma,\sigma'};\ \Delta\ )\ :=\ \Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma'}}(\ \eta^{\sigma,\sigma'}\ )$$

Then, given any $\delta \in \mathbb{C}^c$, *it holds that:*

$$\begin{aligned} D\mathscr{S}^{\sigma} \Big[\ & K,\ \beta,\ S^{\sigma,\sigma'},\ S^{\sigma,\sigma''}; \\ & \Delta[\delta],\ \delta,\ \Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma'}}(\eta^{\sigma,\sigma'})[\delta],\ \Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma''}}(\eta^{\sigma,\sigma''})[\delta]\ \Big]\ +\ \eta^{\sigma}[\delta] \equiv\ 0 \end{aligned}$$

where we have denoted

$$\eta \ := \ \left(\ \eta^{\sigma,\sigma'}, \ \eta^{\sigma,\sigma''} \ \right)^{\mathtt{t}}$$

Proof. Recall the convention

$$A^{\sigma,\sigma'}_{K,\beta} := A^{\sigma,\sigma'}_{K,\beta}[0]$$

Given any $\delta \in \mathbb{C}^c$, consider the linearization

$$\mathscr{S}^\sigma \left[K + \Delta[\delta], \beta + \delta, S^{\sigma,\sigma'} + \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta], S^{\sigma,\sigma''} + \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma''}(\eta^{\sigma,\sigma''})[\delta]\right]$$

$$= \ e^{\sigma}_{\mathscr{S}}[\delta] + \begin{pmatrix} A^{\sigma',\sigma'}_{K,\beta} \ \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta] - \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta] \circ T_\omega \ \ A^{\sigma,\sigma}_{K,\beta} \\ \\ A^{\sigma'',\sigma''}_{K,\beta} \ \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma''}(\eta^{\sigma,\sigma''})[\delta] - \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma''}(\eta^{\sigma,\sigma''})[\delta] \circ T_\omega \ \ A^{\sigma,\sigma}_{K,\beta} \end{pmatrix}$$

$$+ \ \eta^{\sigma}_{\mathscr{S},\Delta}[\delta] \ + \ \mathcal{E}_2(\mathscr{S}^\sigma)\left[K, \ \beta, \ S^{\sigma,\sigma'}, \ S^{\sigma,\sigma''}; \right.$$
$$\left. \Delta[\delta], \delta, \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta], \ \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta]\right]$$

where the term

$$\mathcal{E}_2(\mathscr{S}^\sigma)\left[K, \ \beta, \ S^{\sigma,\sigma'}, \ S^{\sigma,\sigma''}; \Delta[\delta], \delta, \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta], \ \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta]\right]$$

is of higher order.

Thus, we see that

$$D\mathscr{S}^\sigma \left[\ K, \ \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''}; \ \Delta[\delta], \ \delta, \ \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta], \ \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma''}(\eta^{\sigma,\sigma''})[\delta] \ \right]$$

$$= \begin{pmatrix} A^{\sigma',\sigma'}_{K,\beta} \ \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta] - \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma'}(\eta^{\sigma,\sigma'})[\delta] \circ T_\omega \ \ A^{\sigma,\sigma}_{K,\beta} \\ \\ A^{\sigma'',\sigma''}_{K,\beta} \ \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma''}(\eta^{\sigma,\sigma''})[\delta] - \Gamma^{\Delta}_{\mathscr{S}\sigma,\sigma''}(\eta^{\sigma,\sigma''})[\delta] \circ T_\omega \ \ A^{\sigma,\sigma}_{K,\beta} \end{pmatrix} \ + \ \eta^{\sigma}_{\mathscr{S},\Delta}[\delta]$$

Using Lemma 4 and the definition of the operators $\Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma'}}$, we have

$$\begin{pmatrix} A^{\sigma',\sigma'}_{K,\beta}\ \Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma'}}(\eta^{\sigma,\sigma'}) - \Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma'}}(\eta^{\sigma,\sigma'}) \circ T_{\omega}\ \ A^{\sigma,\sigma}_{K,\beta} \\ \\ A^{\sigma'',\sigma''}_{K,\beta}\ \Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma''}}(\eta^{\sigma,\sigma''}) - \Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma''}}(\eta^{\sigma,\sigma''}) \circ T_{\omega}\ \ A^{\sigma,\sigma}_{K,\beta} \end{pmatrix} = - \begin{pmatrix} \eta^{\sigma,\sigma'} + \eta^{\sigma,\sigma'}_{\mathscr{S},\Delta} \\ \\ \eta^{\sigma,\sigma''} + \eta^{\sigma,\sigma''}_{\mathscr{S},\Delta} \end{pmatrix}$$

and so

$$D\mathscr{S}^{\sigma}\Big[\ K,\ \beta, S^{\sigma,\sigma'}, S^{\sigma,\sigma''};\ \Delta[\delta],\ \delta,\ \Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma'}}(\eta^{\sigma,\sigma'})[\delta],\ \Gamma^{\Delta}_{\mathscr{S}^{\sigma,\sigma''}}(\eta^{\sigma,\sigma''})[\delta]\ \Big]$$

$$= \quad - \ \big(\ \eta^{\sigma}[\delta]\ +\ \eta^{\sigma}_{\mathscr{S},\Delta}[\delta]\ \big)\ +\ \ \eta^{\sigma}_{\mathscr{S},\Delta}[\delta]$$

and from here the conclusion of Lemma 8 follows. $\square$

6.3 Approximate right inverse for $D\mathscr{R}$

In this Section we prove Lemma 9, where an approximate right inverse, in the sense of Zehnder, is constructed for $D\mathscr{R}$. We begin by finding $D\mathscr{R}$:

6.3.1 Linearization of $\mathscr{R}$

We now find the effect of small perturbations of $K,\ \beta,\ S,\ U$ on $\mathscr{R}$. The algorithm will choose these corrections to improve the reducibility.

Proposition 2. *Let $\rho > 0,\ K,\ \Delta \in \mathcal{A}_{\rho}^{\mathbb{C}^n},\ \beta,\ \delta \in \mathbb{C}^c$, and $S^{\sigma},\ \chi^{\sigma} \in \mathcal{A}_{\rho,0}^{\mathcal{L}(E_0^{\sigma}, E_0^{\sigma' \oplus \sigma''})}$, where $\sigma,\ \sigma', \sigma''$ is a permutation of the symbols $\{s, u, c\}$. , Assume that $E_S^{\sigma} \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^{\sigma})$. Let $U,\ W \in \mathcal{A}_{\rho}^{\mathbb{C}^{c\times c}}$, $\psi_0 \in \mathcal{A}_{\rho,0}^{\mathcal{L}\left(\mathbb{C}^c \times \mathbb{T}_{\rho}^d,\ E_0^c\right)}$, $\omega \in \mathbb{T}^d$ and $\Lambda \in \mathbb{C}^{c\times c}$.*

Then, it holds that

$$D\mathscr{R}[K,\ \beta,\ S,\ U;\ \Delta,\ \delta,\ \chi,\ W] = \overline{A_{K,\beta}^{c,c}}[S]W - W \circ T_{\omega}\ \Lambda + \eta_{\mathscr{R}}^{\daleth}$$

where we have denoted

$$\begin{aligned} \eta_{\mathscr{R}}^{\daleth} \ :=\ & \left(\psi_0^{-1}\ \ \Pi_{\daleth}^{c}\right) \circ T_{\omega}\ \ [DA_{K,\beta}\ \Delta + \partial_{\beta} A_{K,\beta}\ \delta]\quad \psi_0\ U \\ & - \left(\psi_0^{-1}\ \Pi_{\daleth}^{c}\ \begin{pmatrix} 0 & Z_{\Xi} \\ 0 & 0 \end{pmatrix}\ \Pi_{\daleth}^{c}\right) \circ T_{\omega}\quad A_{K,\beta}\quad \psi_0\ U \end{aligned}$$

where

- *The projection $\Pi_{\daleth}^{c}$ is associated to the splitting defined by*

$$E_{\daleth} = (\ E_S^s\ ,\ E_S^u\ ,\ E_0^c\ )$$

- *We have denoted by Z_Ξ the matrix*

$$
\begin{aligned}
Z_\Xi \;=\; & \chi^{s,c}\,\Pi_0^s \begin{pmatrix} Id_{E_0^s} & S^{u,s} \\ S^{s,u} & Id_{E_0^u} \end{pmatrix}^{-1} \;+\; \chi^{u,c}\,\Pi_0^u \begin{pmatrix} Id_{E_0^s} & S^{u,s} \\ S^{s,u} & Id_{E_0^u} \end{pmatrix}^{-1} \\
& +\; (S^{s,c}\,\Pi_0^s + S^{u,c}\,\Pi_0^u) \begin{pmatrix} Id_{E_0^s} & S^{u,s} \\ S^{s,u} & Id_{E_0^u} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \chi^{u,s} \\ \chi^{s,u} & 0 \end{pmatrix} \begin{pmatrix} Id_{E_0^s} & S^{u,s} \\ S^{s,u} & Id_{E_0^u} \end{pmatrix}^{-1}
\end{aligned}
\tag{107}
$$

Proof. By the definition of $\mathscr{R}$, we have:

$$
\begin{aligned}
& \mathscr{R}[K+\Delta,\ \beta+\delta,\ S+\chi,\ U+W\ ] \\
&= \overline{A^{c,c}_{K+\Delta,\ \beta+\delta}}[S+\chi]\,(U+W) - (U+W)\circ T_\omega\ \Lambda \\
&= \left(\ \psi^{-1}_{S+\chi}\ \Pi^c_{S+\chi}\right)\circ T_\omega\ \ A_{K+\Delta,\beta+\delta}\ \ \psi_{S+\chi}\,(U+W) \\
&\quad - (U+W)\circ T_\omega\ \Lambda
\end{aligned}
\tag{108}
$$

Recall from Subsection 4.1.3 that

$$\psi_{S+\chi} := \left(\ \Pi_0^c\Big|_{E^c_{S+\chi}}\ \right)^{-1}\ \psi_0$$

We now linearize each of the terms in expression (108) by separating the higher order terms resulting from the products. Let us first simplify the factor $\psi^{-1}_{S+\chi}\ \Pi^c_{S+\chi}$:

Let E_Ξ be the splitting defined by

$$E_\Xi = \left(\ E^s_{S+\chi}\ ,\ E^u_{S+\chi}\ ,\ E^c_0\ \right)$$

(see figure 2). It is easy to check that

$$\Pi^c_\Xi = \left(\Pi^c_0\Big|_{E^c_{S+\chi}}\right)\ \ \Pi^c_{S+\chi}$$

Indeed, we have

$$\left(\Pi_0^c\Big|_{E^c_{S+\chi}} \right) \ \Pi^c_{S+\chi}\ v = \begin{cases} v & \text{if } v \in E_0^c \\ \\ 0 & \text{if } v \in E^{s\oplus u}_{S+\chi} \end{cases}$$

which characterizes Π^c_Ξ. Hence,

$$\begin{aligned} \psi^{-1}_{S+\chi}\ \Pi^c_{S+\chi} &= \psi_0^{-1} \left(\Pi_0^c\Big|_{E^c_{S+\chi}} \right) \ \Pi^c_{S+\chi} \\ &= \psi_0^{-1}\ \Pi^c_\Xi \end{aligned} \tag{109}$$

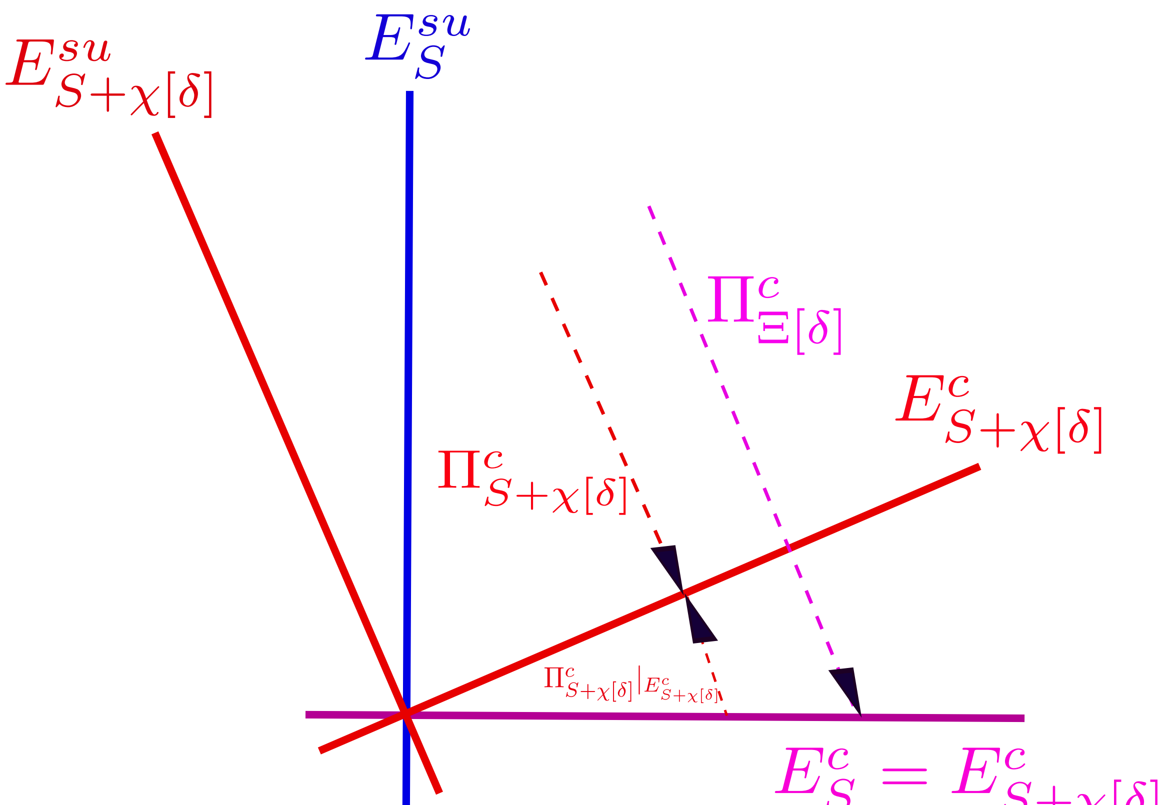

Figure 2: The splitting E_Ξ and its projections.

Using (109) we can rewrite expression (108) for the linearization as

$$\begin{aligned} &\mathscr{R}[K+\Delta,\ \beta+\delta,\ S+\chi,\ U+W\ ] \\ &\quad = \left(\psi_0^{-1}\ \Pi^c_\Xi \right) \circ T_\omega\ \ A_{K+\Delta,\beta+\delta}\ \left(\Pi_0^c|_{E^c_{S+\chi}} \right)^{-1}\ \psi_0\ (U+W) \\ &\qquad - (U+W)\circ T_\omega\ \Lambda \end{aligned} \tag{110}$$

In Section3.1.11 we had found expressions for the matrices of the projections Π^c_{Ξ}, $\Pi^c_{\gimel}$ with respect to the coordinates adapted to the splitting $(E^c_0,\ E^s_0 \oplus E^u_0)$, which are:

$$\Pi^c_{\Xi} = \begin{pmatrix} \mathrm{Id}_{E^c_0} & \Phi^{E^s_0 \oplus E^u_0, E^c_0}_{\mathcal{V}^{s+u}_\rho}(E^s_{S+\chi} \oplus E^u_{S+\chi}) \\ & \\ 0 & \mathrm{Id}_{E^{s\oplus u}_0} \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{Id}_{E^c_0} \\ \\ 0 \end{pmatrix}, \tag{111}$$

$$\Pi^c_{\gimel} = \begin{pmatrix} \mathrm{Id}_{E^c_0} & \Phi^{E^s_0 \oplus E^u_0, E^c_0}_{\mathcal{V}^{s+u}_\rho}(E^s_{S} \oplus E^u_{S}) \\ & \\ 0 & \mathrm{Id}_{E^{s\oplus u}_0} \end{pmatrix}^{-1} \begin{pmatrix} \mathrm{Id}_{E^c_0} \\ \\ 0 \end{pmatrix} \tag{112}$$

where the expressions for $\Phi^{E^s_0 \oplus E^u_0, E^c_0}_{\mathcal{V}^{s+u}_\rho}(E^s_{S+\chi} \oplus E^u_{S+\chi})$ and $\Phi^{E^s_0 \oplus E^u_0, E^c_0}_{\mathcal{V}^{s+u}_\rho}(E^s_{S} \oplus E^u_{S})$ had been computed in Subsection 3.1.10:

$$\begin{aligned} \Phi^{E^s_0 \oplus E^u_0, E^c_0}_{\mathcal{V}^{s+u}_\rho}(E^s_{S+\chi} \oplus E^u_{S+\chi}) \ &= \ (S^{s,c} + \chi^{s,c})\ \Pi^s_0 \begin{pmatrix} \mathrm{Id}_{E^s_0} & S^{u,s} + \chi^{u,s} \\ S^{s,u} + \chi^{s,u} & \mathrm{Id}_{E^u_0} \end{pmatrix}^{-1} \\ &+ \ (S^{u,c} + \chi^{u,c})\ \Pi^u_0 \begin{pmatrix} \mathrm{Id}_{E^s_0} & S^{u,s} + \chi^{u,s} \\ S^{s,u} + \chi^{s,u} & \mathrm{Id}_{E^u_0} \end{pmatrix}^{-1} \end{aligned}$$

Expanding a Neumann series for each of the two matrix inverses which appear in the expression for $\Phi^{E_0^s \oplus E_0^u, E_0^c}_{\mathcal{V}_\rho^{s+u}}(E^s_{S+\chi} \oplus E^u_{S+\chi})$, we obtain:

$$
\begin{aligned}
\Phi^{E_0^s \oplus E_0^u, E_0^c}_{\mathcal{V}_\rho^{s+u}}(E^s_{S+\chi} \oplus E^u_{S+\chi}) \; &= \Phi^{E_0^s \oplus E_0^u, E_0^c}_{\mathcal{V}_\rho^{s+u}}(E^s_{S} \oplus E^u_{S}) \\
&+ \; \chi^{s,c}\, \Pi_0^s \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1} \; + \; \chi^{u,c}\, \Pi_0^u \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1} \\
&+ \; (S^{s,c}\, \Pi_0^s + S^{u,c}\, \Pi_0^u) \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \chi^{u,s} \\ \chi^{s,u} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1} \\
&+ \mathcal{E}_2 \Big(\; \Phi^{E_0^s \oplus E_0^u, E_0^c}_{\mathcal{V}_\rho^{s+u}}(E^s_{S+\chi} \oplus E^u_{S+\chi}) \; \Big)
\end{aligned}
$$

where $\mathcal{E}_2 \Big(\; \Phi^{E_0^s \oplus E_0^u, E_0^c}_{\mathcal{V}_\rho^{s+u}}(E^s_{S+\chi} \oplus E^u_{S+\chi}) \; \Big)$ is of higher order.

Expanding the Neumann series for the matrix inverse which appears in expression (111) for Π^c_Ξ, we obtain

$$
\Pi^c_\Xi = \; \Pi^c_\daleth \; + \; \Pi^c_\daleth \begin{pmatrix} 0 & Z_\Xi \\ 0 & 0 \end{pmatrix} \Pi^c_\daleth \; + \; \mathcal{E}_2(\Pi^c_\Xi)
$$

where $\mathcal{E}_2(\Pi^c_\Xi)$ is of higher order.

Substituting in (110), we obtain the expression for the linearization of $\mathscr{R}$ given in the statement of Proposition 2.

$\square$

We are now ready to construct an approximate right inverse for $D\mathscr{R}$:

Lemma 9. *Let* $0 < \zeta < \rho$, $K \in \mathcal{A}_\rho^{\mathbb{C}^n}$, β, $\xi \in \mathbb{C}^c$, *and* $S^\sigma \in \mathcal{A}_{\rho,0}^{\mathcal{L}(E_0^\sigma, E_0^{\sigma' \oplus \sigma''})}$, *where* σ, σ', σ'' *is a permutation of the symbols* $\{s, u, c\}$. *Assume that* $E_S^\sigma \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^\sigma)$. *Let* $U \in \mathcal{A}_\rho^{\mathbb{C}^{c \times c}}$ *be such that* $\left| U^{(-1)} \;\right|_\rho \leq C$, $\psi_0 \in \mathcal{A}_{\rho,0}^{\mathcal{L}\left(\mathbb{C}^c \times \mathbb{T}^d_\rho,\; E_0^c\right)}$ *and* $\Lambda \in \mathbb{C}^{c \times c}$, $\omega \in \mathbb{T}^d$,

$\Lambda = diag[\lambda_1,\ \dots,\ \lambda_c]$, *such that*

$$\omega \in DC(\nu, \tau)$$

$$Spec(\Lambda) \in DC^{2nd}_{\omega}(\nu, \tau)$$

Denote the error of the Reducibility equation by $e_{\mathscr{R}} := \mathscr{R}\,[K, \beta, S^c, U]$.

Given affine functions

$$\begin{aligned} \Delta \ :\ \mathbb{C}^c &\to \mathcal{A}^{\mathbb{C}^n}_{\rho} \\ \delta &\mapsto \Delta[\delta]. \end{aligned} \qquad\qquad \begin{aligned} \chi^{\sigma} \ :\ \mathbb{C}^c &\to \mathcal{A}^{\mathcal{L}(E^{\sigma}_0, E^{\sigma' \oplus \sigma''}_0)}_{\rho,0} \\ \delta &\mapsto \chi^{\sigma}[\delta], \end{aligned}$$

and let

$$\chi \ :=\ \prod_{\sigma \neq \sigma'} \chi^{\sigma,\sigma'}$$

we introduce the affine function of δ *taking values in* $\mathcal{A}^{\mathbb{C}^{c\times c}}_{\rho}$*:*

$$\begin{aligned} \eta^{\daleth}_{\mathscr{R}}[\delta] \ :=\ & \left(\psi_0^{-1}\ \ \Pi^{c}_{\daleth}\right) \circ T_{\omega}\ \ [DA_{K,\beta}\ \Delta[\delta] + \partial_{\beta} A_{K,\beta}\ \delta]\quad \psi_0\ U \\ & - \left(\psi_0^{-1}\ \Pi^{c}_{\daleth}\ \begin{pmatrix} 0 & Z_{\Xi}[\delta] \\ & \\ 0 & 0 \end{pmatrix}\ \Pi^{c}_{\daleth}\right) \circ T_{\omega}\quad A_{K,\beta}\quad \psi_0\ U \end{aligned}$$

where The projection $\Pi^{c}_{\daleth}$ *is associated to the splitting*

$$E_{\daleth} = (\ E^s_S\ ,\ E^u_S\ ,\ E^c_0\ ),$$

and we have denoted by $Z_{\Xi}[\delta]$ *the matrix*

$$\begin{aligned} Z_{\Xi}[\delta] \ =\ & \chi^{s,c}[\delta]\ \Pi^s_0\ \begin{pmatrix} Id_{E^s_0} & S^{u,s} \\ & \\ S^{s,u} & Id_{E^u_0} \end{pmatrix}^{-1} \ +\ \chi^{u,c}[\delta]\ \Pi^u_0\ \begin{pmatrix} Id_{E^s_0} & S^{u,s} \\ & \\ S^{s,u} & Id_{E^u_0} \end{pmatrix}^{-1} \\ & +\ (S^{s,c}\ \Pi^s_0 + S^{u,c}\ \Pi^u_0)\ \ \begin{pmatrix} Id_{E^s_0} & S^{u,s} \\ & \\ S^{s,u} & Id_{E^u_0} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \chi^{u,s}[\delta] \\ & \\ \chi^{s,u}[\delta] & 0 \end{pmatrix} \begin{pmatrix} Id_{E^s_0} & S^{u,s} \\ & \\ S^{s,u} & Id_{E^u_0} \end{pmatrix}^{-1} \end{aligned}$$

Recall from Remark 23 the operator $\left(\widetilde{\Gamma}_{\mathscr{R},\xi},\ \tilde{\gamma}_{\mathscr{R}} \right)$*, which acts on affine functions and takes values in* $\mathcal{A}_{\rho-\zeta}^{\mathbb{C}^{c\times c}} \times \mathbb{C}^c$*. Let* $(\ \Gamma_{\mathscr{R},\xi},\ \gamma_{\mathscr{R}};\ \Delta,\ \chi\ )$ *be the operator acting on affine functions and taking values in* $\mathcal{A}_{\rho-\zeta}^{\mathbb{C}^{c\times c}} \times \mathbb{C}^c$ *defined as follows: Given an affine function*

$$\eta\ :\ \mathbb{C}^c \to \mathcal{A}_{\rho}^{\mathbb{C}^{c\times c}}$$

$$\delta \mapsto \eta[\delta],$$

Let

$$(\ \Gamma_{\mathscr{R},\xi}(\eta),\ \gamma_{\mathscr{R}}(\eta);\ \Delta,\ \chi\ )\ :=\ \left(\ \widetilde{\Gamma}_{\mathscr{R},\xi}(\eta + \eta^{\daleth}_{\mathscr{R}}),\ \tilde{\gamma}_{\mathscr{R}}(\eta + \eta^{\daleth}_{\mathscr{R}})\ \right)$$

When typographically convenient, we will denote

$$(\ \Gamma_{\mathscr{R},\xi}(\eta),\ \gamma_{\mathscr{R}}(\eta);\ \Delta,\ \chi\ )\ :=\ \left(\ \Gamma^{\Delta,\ \chi}_{\mathscr{R},\xi}(\eta),\ \gamma^{\Delta,\ \chi}_{\mathscr{R}}(\eta)\ \right)$$

Assume also that the following non-degeneracy condition holds:

$$M := \left(\min_{1\le j\le c} \left|\ \widehat{\left[\ \mathtt{Lin}\left(\ U^{(-1)} \circ T_{\omega}\ \eta^{\daleth}_{\mathscr{R}}\ \right)\ \right]}_{jj}(0)\ \right| \right)^{-1} < \infty \tag{113}$$

Then, the following estimate holds:

$$\begin{aligned} &\Bigg|\ D\mathscr{R}\left[K, \beta, S, U;\ \Delta[\gamma^{\Delta,\chi}_{\mathscr{R}}(\eta)],\ \gamma^{\Delta,\chi}_{\mathscr{R}}(\eta),\ \chi[\gamma^{\Delta,\chi}_{\mathscr{R}}(\eta)], \Gamma^{\Delta,\chi}_{\mathscr{R},\xi}(\eta)\right] \\ &\qquad + \eta[\gamma^{\Delta,\chi}_{\mathscr{R}}(\eta)]\ \Bigg|_{\rho-\zeta} \\ &\le \frac{C\ M}{\zeta^{\tau}}\ \left|\ e_{\mathscr{R}}\ \right|_{\rho}\ \left|\ \mathtt{Cte}(\eta + \eta^{\daleth}_{\mathscr{R}})\ \right|_{\rho}\ \left|\ \mathtt{Lin}(\eta + \eta^{\daleth}_{\mathscr{R}})\ \right|_{\rho} \end{aligned} \tag{114}$$

Proof. From Proposition 2 it follows that

$$D\mathscr{R}\left[K, \beta, S, U;\ \Delta[\gamma_{\mathscr{R}}^{\Delta,\chi}(\eta)],\ \gamma_{\mathscr{R}}^{\Delta,\chi}(\eta),\ \chi[\gamma_{\mathscr{R}}^{\Delta,\chi}(\eta)], \Gamma_{\mathscr{R},\xi}^{\Delta,\chi}(\eta)\right] + \eta[\gamma_{\mathscr{R}}^{\Delta,\chi}(\eta)]$$

$$= \overline{A_{K,\beta}^{c,c}}[S]\ \Gamma_{\mathscr{R},\xi}^{\Delta,\chi}(\eta) - \Gamma_{\mathscr{R},\xi}^{\Delta,\chi}(\eta) \circ T_\omega + \eta_{\mathscr{R}}^{\gimel}[\gamma_{\mathscr{R}}^{\Delta,\chi}(\eta)] + \eta[\gamma_{\mathscr{R}}^{\Delta,\chi}(\eta)]$$

By Lemma 6 and the definition of the operator $\Gamma_{\mathscr{R},\xi}$ we have

$$\overline{A_{K,\beta}^{c,c}}[S]\ \Gamma_{\mathscr{R},\xi}^{\Delta,\chi}(\eta) - \Gamma_{\mathscr{R},\xi}^{\Delta,\chi}(\eta) \circ T_\omega + \eta_{\mathscr{R}}^{\gimel}[\gamma_{\mathscr{R}}^{\Delta,\chi}(\eta)] + \eta[\gamma_{\mathscr{R}}^{\Delta,\chi}(\eta)] \ = \ e_{qNwt}^{\mathscr{R}}$$

Estimate (114) follows now from estimate (88). $\square$

6.4 Non-linear estimates.

In this section we prove Lemma 10, where we collect the estimates for the additive corrections $\Delta[\gamma]$, γ, $\chi[\gamma]$, $W[\gamma]$ (estimates (115) - (118) below). We then estimate the errors of the improved solutions of the non-linear invariance equations (estimates (120), (122), (123)), given that the geometric conditions (119), (121) are satisfied. Finally, estimates for the deterioration of the hyperbolicity constant of the corrected splitting are given in (124).

Lemma 10. *Let* $0 < \zeta < \rho$, $K \in \mathcal{A}_\rho^{\mathbb{C}^n}$, $\beta,\ \chi \in \mathbb{C}^c$, $S^\sigma \in \mathcal{A}_{\rho,0}^{\mathcal{L}(E_0^\sigma, E_0^{\sigma'\oplus\sigma''})}$, $\sigma \in \{s,u,c\}$, *where* $\{\sigma, \sigma', \sigma''\}$ *is a permutation of the symbols* $\{s,u,c\}$*. Let* $U \in \mathcal{A}_\rho^{GL(\mathbb{C}^c)}$ *be such that* $\left|U^{(-1)}\right|_\rho \leq C$*. Assume that the splitting* E_S *is* $\mathtt{N}$, C_h, $\mu_{K,\beta}^{S,s}$, $\mu_{K,\beta}^{S,u}$, $\mu_{K,\beta}^{S,c,+}$, $\mu_{K,\beta}^{S,c,-}$*-whiskered with respect to* $A_{K,\beta}$*, and that the splitting* E_0 *is* $\mathtt{N}$, C_h, $\mu_{K,\beta}^{0,s}$, $\mu_{K,\beta}^{0,u}$, $\mu_{K,\beta}^{0,c,+}$, $\mu_{K,\beta}^{0,c,-}$*-whiskered with respect to* $A_{K,\beta}$*. Let* $\omega \in \mathbb{T}^d$*, and* $\Lambda \in \mathbb{C}^{c\times c}$, $\Lambda = diag[\lambda_1,\ \ldots,\ \lambda_c]$ *be such that*

$$\omega \in DC(\nu,\tau),$$

$$Spec(\Lambda) \in DC_\omega^{1st}(\nu,\tau)\ \cap DC_\omega^{2nd}(\nu,\tau)$$

Assume that $K(\mathtt{T}_\rho^d) \subset \mathcal{D}_{\rho_0,\eta_0}(K_0,\beta_0)$ *and that* $E_S^\sigma \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E_0^\sigma)$*. We denote the errors of the invariance equation for the torus, the invariance equation for the splitting and the reducibility equation by*

$$e := \mathscr{I}[K,\beta],$$

$$\mathscr{S}^\sigma[K,\ \beta,\ S^{\sigma,\sigma'},\ S^{\sigma,\sigma''}]\ :=\ e_{\mathscr{S}}^\sigma\ :=\ \left(e_{\mathscr{S}}^{\sigma,\sigma'}, e_{\mathscr{S}}^{\sigma,\sigma''}\right)^{\mathtt{t}},$$

$$e_{\mathscr{R}} := \mathscr{R}\,[K,\beta,S,U]$$

Define the affine functions:

$$\Delta \;:=\; \Gamma_{\mathscr{I}}(-e)\;,$$

$$\chi^{\sigma,\sigma'} \;:=\; \Gamma_{\mathscr{S}^{\sigma,\sigma'}}(-e_{\mathscr{S}}^{\sigma,\sigma'};\;\Delta),\;\; \sigma\neq\sigma'\;,\quad \chi \;:=\; \prod_{\sigma\neq\sigma'} \chi^{\sigma,\sigma'}$$

Assume that the following non-degeneracy condition holds:

$$M := \left(\min_{1\le j\le c} \Big| \; \Big[\; \mathtt{Lin}\,\big(\; U^{(-1)} \circ T_\omega \; \eta^{\daleth}_{\mathscr{R}} \;\big) \; \Big]^{\widehat{}}_{jj}\,(0) \; \Big| \; \right)^{-1} < \infty$$

where the expression for the affine function $\eta^{\daleth}_{\mathscr{R}}$ appears in Lemma 9.

Finally, let

$$(W,\;\gamma\;) \;:=\; (\;\Gamma_{\mathscr{R},\;\xi}(e_{\mathscr{R}}),\;\gamma_{\mathscr{R}}(e_{\mathscr{R}});\;\Delta,\;\chi\;)$$

Then, the following estimates hold

$$|\;\gamma\;| \;\le\; \frac{C\;M}{\zeta^\tau}\;\Big(\;|\;e_{\mathscr{R}}\;|_\rho + |\;e_{\mathscr{S}}\;|_\rho + |\;e\;|_\rho\Big), \tag{115}$$

$$|\;\Delta[\gamma]\;|_{\rho-\zeta} \;\le\; \frac{C\;M}{\zeta^{2\tau}}\;\Big(\;|\;e_{\mathscr{R}}\;|_\rho + |\;e_{\mathscr{S}}\;|_\rho + |\;e\;|_\rho\Big)\;, \tag{116}$$

$$|\;\chi[\gamma]\;|_{\rho-\zeta} \;\;\le\; \frac{C\;M}{\zeta^{2\tau}}\;\Big(\;|\;e_{\mathscr{R}}\;|_\rho + |\;e_{\mathscr{S}}\;|_\rho + |\;e\;|_\rho\Big)\;, \tag{117}$$

$$|\;W\;|_{\rho-2\zeta} \;\le\; \frac{C\;M}{\zeta^{2\tau}}\;\Big(\;|\;e_{\mathscr{R}}\;|_\rho + |\;e_{\mathscr{S}}\;|_\rho + |\;e\;|_\rho\Big) \tag{118}$$

Furthermore: If $|\gamma|$ is small enough that

$$K + \Delta[\gamma]\,\big(\;\mathbb{T}^d_\rho\;\big) \;\subset\; \mathcal{D}_{\rho_0,2\eta_0}(K_0,\;\beta_0) \tag{119}$$

then the following estimate holds:

$$|\;\mathscr{I}[K+\Delta[\gamma],\;\beta+\gamma]\;|_{\rho-\zeta} \;\le \frac{C\;M}{\zeta^{2\tau}}\Big(\;|\;e_{\mathscr{R}}\;|_\rho + |\;e_{\mathscr{S}}\;|_\rho + |\;e\;|_\rho\Big)^2 \tag{120}$$

If $|\gamma|$ is small enough that

$$E^\sigma_{S+\chi[\gamma]} \in \mathcal{V}_{\eta_0^{\mathscr{S}}}(E^\sigma_0)\;,\quad \sigma\in\{s,u,c\} \tag{121}$$

then the following estimates hold:

$$\begin{aligned} &\Big| \mathscr{S}^{\sigma} \Big[\; K+\Delta[\gamma], \; \beta+\gamma, \; S^{\sigma,\sigma'}+\chi^{\sigma,\sigma'}[\gamma], \; S^{\sigma,\sigma''}+\chi^{\sigma,\sigma''}[\gamma] \; \Big] \; \Big|_{\rho-\zeta} \\ &\quad \leq \; \frac{C \; M^2}{\zeta^{4\tau}} \; \Big(\mid e_{\mathscr{R}} \mid_{\rho} + \mid e_{\mathscr{S}} \mid_{\rho} + \mid e \mid_{\rho} \Big)^2 \end{aligned} \tag{122}$$

$$\begin{aligned} &\mid \mathscr{R}\,[K+\Delta[\gamma], \beta+\gamma, S+\chi[\gamma], U+W] \mid_{\rho-2\zeta} \\ &\quad \leq \frac{C \; M^2}{\zeta^{4\tau}} \; \Big(\mid e_{\mathscr{R}} \mid_{\rho} + \mid e_{\mathscr{S}} \mid_{\rho} + \mid e \mid_{\rho} \Big)^2 \end{aligned} \tag{123}$$

Finally, we can take [2]

$$\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma} > \mu^{0,\sigma}_{K,\beta} \; , \quad \mu^{S+\chi[\gamma],\sigma}_{K+\Delta[\gamma],\beta+\gamma} > \mu^{S,\sigma}_{K,\beta}$$

$$C_h(A_{K+\Delta[\gamma],\beta+\gamma}, \; E_{S+\chi[\gamma]}) > C_h(A_{K,\beta}, \; E_S)$$

such that the following estimates hold:

$$\begin{aligned} \max\Big\{ \; &\mu^{S+\chi[\gamma],\sigma}_{K+\Delta[\gamma],\beta+\gamma} - \mu^{S,\sigma}_{K,\beta}, \; \mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma} - \mu^{0,\sigma}_{K,\beta}, \\ &C_h(A_{K+\Delta[\gamma],\beta+\gamma}, \; E_{S+\chi[\gamma]}) - C_h(A_{K,\beta}, \; E_S) \; \Big\} \\ &\leq \; \frac{M}{\zeta^{2\tau}} \; \Big(\mid e_{\mathscr{R}} \mid_{\rho} + \mid e_{\mathscr{S}} \mid_{\rho} + \mid e \mid_{\rho} \Big) \end{aligned} \tag{124}$$

Proof. We first use inequality (87) to estimate $\mid \gamma \mid$. We have

$$\mid \gamma \mid \; \leq \; C \; M \; \Big| \, \mathtt{Cte}(e_{\mathscr{R}} + \eta^{\daleth}_{\mathscr{R}}) \, \Big|_{\rho-\zeta} \tag{125}$$

Note that

$$\begin{aligned} \mathtt{Cte}(e_{\mathscr{R}} + \eta^{\daleth}_{\mathscr{R}}) \; = \; & e_{\mathscr{R}} + \big(\psi_0^{-1} \;\; \Pi^c_{\daleth}\big) \circ T_\omega \;\; [DA_{K,\beta} \; \mathtt{Cte}(\Delta)] \quad \psi_0 \; U \\ & - \left(\psi_0^{-1} \; \Pi^c_{\daleth} \begin{pmatrix} 0 & \mathtt{Cte}(Z_{\Xi}) \\ 0 & 0 \end{pmatrix} \Pi^c_{\daleth} \right) \circ T_\omega \quad A_{K,\beta} \quad \psi_0 \; U \end{aligned}$$

---

[2] *Recall from equation (22) the definition of the (deteriorated) hyperbolicity constants:* $\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}$, $\mu^{S+\chi[\gamma],\sigma}_{K+\Delta[\gamma],\beta+\gamma}$, $C_h(A_{K+\Delta[\gamma],\beta+\gamma}, \; E_{S+\chi[\gamma]})$.

where

$$\mathtt{Cte}(Z_\Xi) \;=\; \mathtt{Cte}(\chi^{s,c})\ \Pi_0^s \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1} + \mathtt{Cte}(\chi^{u,c})\ \Pi_0^u \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1}$$

$$+ (S^{s,c}\Pi_0^s + S^{u,c}\Pi_0^u) \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \mathtt{Cte}(\chi^{u,s}) \\ \mathtt{Cte}(\chi^{s,u}) & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{E_0^s} & S^{u,s} \\ S^{s,u} & \mathrm{Id}_{E_0^u} \end{pmatrix}^{-1}$$

Using estimates (35), (57), and the definition of Δ, we obtain

$$|\ \mathtt{Cte}(\Delta)\ |_{\rho-\zeta} \;\le\; \frac{C}{\zeta^\tau}\ |\ e\ |_\rho \tag{126}$$

Using estimates (64), and the definition of χ, we have

$$|\ \mathtt{Cte}(\chi)\ |_{\rho-\zeta} \;\le\; C\ \Big(\ |\ e_{\mathscr{S}}\ |_{\rho-\zeta} + |\ \mathtt{Cte}(\Delta)\ |_{\rho-\zeta}\ \Big) \tag{127}$$

Hence, using estimate (126),

$$|\ \mathtt{Cte}(\chi)\ |_{\rho-\zeta} \;\le\; \frac{C}{\zeta^\tau}\ \Big(\ |\ e_{\mathscr{S}}\ |_\rho + |\ e\ |_\rho\ \Big) \tag{128}$$

Using estimates (126), (128) and the expression for $\mathtt{Cte}(e_{\mathscr{R}} + \eta^{\daleth}_{\mathscr{R}})$ above, we obtain

$$\Big|\ \mathtt{Cte}(e_{\mathscr{R}} + \eta^{\daleth}_{\mathscr{R}})\ \Big|_{\rho-\zeta} \;\le\; \frac{C}{\zeta^\tau}\ \Big(\ |\ e_{\mathscr{R}}\ |_\rho + |\ e_{\mathscr{S}}\ |_\rho + |\ e\ |_\rho\ \Big) \tag{129}$$

Finally, using estimate (129) and (125) we obtain estimate (115) for $|\ \gamma\ |$.

Using the Taylor expansion of $\mathscr{I}$ up to first order we have

$$\begin{aligned} |\ \mathscr{I}[K+\Delta[\gamma],\ \beta+\gamma]\ |_{\rho-\zeta} \;\le\; &|\ e + D\mathscr{I}\,[\ K,\ \beta;\ \Delta[\gamma],\ \gamma\ ]\ |_{\rho-\zeta} \\ &+ |\ \mathcal{E}_2(\mathscr{I})\,[(K,\beta);\ \Delta[\gamma],\ \gamma]\ |_{\rho-\zeta} \end{aligned} \tag{130}$$

Using estimate (95), and the definition of the affine function Δ, we obtain the estimate

$$
\begin{aligned}
& |\ e\ +\ D\mathscr{I}\,[\ K,\ \beta;\ \Delta[\gamma],\ \gamma\ ]\ |_{\rho-\zeta} \\
& \quad \leq\ C\ \ (\ |\ e\ |_{\rho}\ +\ |\ \partial_{\beta}\ f_{\beta}\circ K\ |_{\rho}\ |\ \gamma\ |\ )\ \frac{|\ e_{\mathscr{S}}\ |_{\rho}\ +\ |\ e_{\mathscr{R}}\ |_{\rho}}{\zeta^{\tau}} \\
& \quad \leq\ \frac{C\ M}{\zeta^{2\tau}}\ \Big(\ |\ e_{\mathscr{R}}\ |_{\rho}+|\ e_{\mathscr{S}}\ |_{\rho}+|\ e\ |_{\rho}\ \Big)^{2}
\end{aligned}
\tag{131}
$$

By assumption (119) we have

$$
\operatorname{dist}\Big(\ (\ K+\Delta\ )\,(\mathbb{T}^{d}_{\rho})\ ,\operatorname{Domain}(f)\ \Big)>\eta_{0}
$$

Using the Mean Value Theorem and Taylor's estimate we then have

$$
\begin{aligned}
|\ \mathcal{E}_{2}(\mathscr{I})\,[(K,\beta);\ \Delta[\gamma],\ \gamma]\ |_{\rho-\zeta}\ & \leq \Big(\ |\ \Delta[\gamma]\ |_{\rho-\zeta}+|\ \gamma\ |\ \Big)^{2}\ (\ |\ D^{2}f\ |_{\rho}+|\ \partial^{2}_{\beta}f\ |\ ) \\
& \leq C\ \ \Big(\ |\ \Delta[\gamma]\ |_{\rho-\zeta}+|\ \gamma\ |\ \Big)^{2}
\end{aligned}
\tag{132}
$$

We estimate $|\ \Delta[\gamma]\ |_{\rho-\zeta}$ as

$$
|\ \Delta[\gamma]\ |_{\rho-\zeta}\ \leq\ |\ \texttt{Cte}(\Delta)\ |_{\rho-\zeta}+|\ \texttt{Lin}(\Delta)\ |_{\rho-\zeta}\ \ |\ \gamma\ |
$$

An estimate for $|\ \texttt{Cte}(\Delta)\ |_{\rho-\zeta}$ had been obtained in (126). Using estimates (57) and the definition of Δ we have

$$
|\ \texttt{Lin}(\Delta)\ |_{\rho-\zeta}\ \leq\ \frac{C}{\zeta^{\tau}}
\tag{133}
$$

Hence, using estimate (115), we obtain estimate (116) for $|\ \Delta[\gamma]\ |_{\rho-\zeta}$.

Substituting (116) in the right hand side of estimate (132) we obtain

$$
|\ \mathcal{E}_{2}(\mathscr{I})\,[K,\beta;\ \Delta[\gamma],\gamma]\ |_{\rho-\zeta}\ \leq\ \frac{C\ M}{\zeta^{2\tau}}\Big(\ |\ e_{\mathscr{R}}\ |_{\rho}\ +\ |\ e_{\mathscr{S}}\ |_{\rho}\ +\ |\ e\ |_{\rho}\ \Big)^{2}
\tag{134}
$$

Substituting estimates (116), (131) in the right-hand side of (130), we obtain estimate (120) in the statement of Lemma 10.

We now estimate $|\ \chi[\gamma]\ |_{\rho-\zeta}$ as

$$|\ \chi[\gamma]\ |_{\rho-\zeta}\ \leq\ |\ \mathtt{Cte}(\chi)\ |_{\rho-\zeta}\ +\ |\ \mathtt{Lin}(\chi)\ |_{\rho-\zeta}\ |\ \gamma\ |$$

An estimate for $|\ \mathtt{Cte}(\chi)\ |_{\rho-\zeta}$ had been obtained in (128). Using estimates (64), (133), and the definition of χ we have

$$\begin{aligned} |\ \mathtt{Lin}(\chi)\ |_{\rho-\zeta}\ &\leq\ C\ |\ \mathtt{Lin}(\Delta)\ |_{\rho} \\ &\leq\ \frac{C}{\zeta^{\tau}} \end{aligned} \tag{135}$$

Hence, using (115), we obtain estimate (117) for $|\ \chi[\gamma]\ |_{\rho-\zeta}$.

By assumption (121), the expression

$$\mathscr{S}^{\sigma}\left[\ K+\Delta[\gamma],\ \beta+\gamma,\ S^{\sigma,\sigma'}+\chi^{\sigma,\sigma'}[\gamma],\ S^{\sigma,\sigma''}+\chi^{\sigma,\sigma''}[\gamma]\ \right]$$

is well-defined. We apply Lemma 8 to obtain

$$\begin{aligned} &\mathscr{S}^{\sigma}\left[\ K+\Delta[\gamma],\ \beta+\gamma,\ S^{\sigma,\sigma'}+\chi^{\sigma,\sigma'}[\gamma],\ S^{\sigma,\sigma''}+\chi^{\sigma,\sigma''}[\gamma]\ \right] \\ &\quad = \mathcal{E}_2(\mathscr{S}^{\sigma})\left[\ K,\beta,S^{\sigma,\sigma'},S^{\sigma,\sigma''};\ \Delta[\gamma],\gamma,\chi^{\sigma,\sigma'}[\gamma],\chi^{\sigma,\sigma''}[\gamma]\ \right] \end{aligned} \tag{136}$$

where $\mathcal{E}_2(\mathscr{S}^{\sigma})$ is of higher order and admits the estimate

$$\begin{aligned} &\left|\ \mathcal{E}_2(\mathscr{S}^{\sigma})\left[\ K,\beta,S^{\sigma,\sigma'},S^{\sigma,\sigma''};\ \Delta[\gamma],\beta+\gamma,\chi^{\sigma,\sigma'}[\gamma],\chi^{\sigma,\sigma''}[\gamma]\ \right]\ \right|_{\rho-\zeta} \\ &\leq\ \frac{C\ M^2}{\zeta^{4\tau}}\ \Big(\ |\ e_{\mathscr{R}}\ |_{\rho}\ +\ |\ e_{\mathscr{S}}\ |_{\rho}\ +\ |\ e\ |_{\rho}\ \Big)^2 \end{aligned} \tag{137}$$

Using (136) and (137), we obtain estimate (122) in the statement of Lemma 10.

By Lemma 9, we have

$$
\begin{aligned}
&\mathscr{R}[\ K+\Delta[\gamma], \beta+\gamma, S+\chi[\gamma], U+W\ ] \\
&\quad = \ e^{\mathscr{R}}_{qNwt} \ + \ \mathcal{E}_2(\mathscr{R})[\ K, \beta, S, U;\ \Delta[\gamma], \gamma, \chi[\gamma], W\ ]
\end{aligned}
\tag{138}
$$

Using estimate (114) from Lemma 9 we obtain estimate (123) in the statement of Lemma 10.

Finally, we estimate the deteriorated geometric constants for the splitting $E_{S+\chi[\gamma]}$ as follows: For $\sigma \in \{s, u, c\}$, denote

$$
\mathcal{E}_{\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} := \ \Pi^{\sigma}_0 \circ T_\omega \ \left(\ DA_{K,\beta}\ \Delta[\gamma] + \partial_\beta A_{K,\beta}\ \gamma + \mathcal{E}_2\,(Df)\ \right)\ \Pi^{\sigma}_0
$$

where we used the shorthand notation $\mathcal{E}_2\,(Df) \ := \ \mathcal{E}_2\,(Df)\ [\ K, \beta;\ \Delta[\gamma], \gamma\ ]$.

Using estimate (116), we have

$$
\left|\ \mathcal{E}_{\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}}\ \right|_{\rho-\zeta} \leq \ \frac{C\ M}{\zeta^{2\tau}}\ \Big(\ |\ e_{\mathscr{R}}\ |_\rho \ + \ |\ e_{\mathscr{S}}\ |_\rho \ + \ |\ e\ |_\rho\ \Big)
\tag{139}
$$

Note that, for $1 \leq j \leq \mathtt{N}$,

$$
\begin{aligned}
\left|\ \Big(\ A^{\sigma,\,\sigma}_{K+\Delta[\gamma],\beta+\gamma}\ \Big)^{(\pm j)}\ \right|_{\rho-\zeta} &= \left|\ (\ \Pi^{\sigma}_0 \circ T_\omega \ \ A_{K+\Delta[\gamma],\beta+\gamma}\ \ \Pi^{\sigma}_0\ )^{(\pm j)}\ \right|_{\rho-\zeta} \\
&\leq \ \left|\ \Big(\ A^{\sigma,\sigma}_{K,\beta}\ + \ \mathcal{E}_{\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}}\ \Big)^{(\pm j)}\ \right|_{\rho-\zeta} \\
&\leq \ \left|\ (\ A^{\sigma,\sigma}_{K,\beta}\ )^{(\pm j)}\ \right|_{\rho-\zeta} \ + \ \left|\ \widetilde{\mathcal{E}}_{\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}}\ \right|_{\rho-\zeta}
\end{aligned}
$$

where

$$
\widetilde{\mathcal{E}}_{\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} \ := \ \Big(\ A^{\sigma,\sigma}_{K,\beta}\ + \ \mathcal{E}_{\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}}\ \Big)^{(\pm j)} \ - \ (\ A^{\sigma,\sigma}_{K,\beta}\ )^{(\pm j)}
$$

is estimated (using the binomial expansion) as

$$
\begin{aligned}
\left|\ \widetilde{\mathcal{E}}_{\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}}\ \right|_{\rho-\zeta} \ &\leq \ C\ \ (\ C_h(A_{K,\beta},\ E_S)\ )^{j-1}\ \left|\ \mathcal{E}_{\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}}\ \right|_{\rho-\zeta} \\
&\leq \ \frac{C\ M}{\zeta^{2\tau}}\ \Big(\ |\ e_{\mathscr{R}}\ |_\rho + |\ e_{\mathscr{S}}\ |_\rho + |\ e\ |_\rho\ \Big)
\end{aligned}
\tag{140}
$$

so that we can take

$$\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma} \;\leq\; \mu^{0,\sigma}_{K,\beta} + \Big|\; \widetilde{\mathcal{E}}_{\mu^{0,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} \;\Big|_{\rho-\zeta}\,,$$

$$C_h(A_{K+\Delta[\gamma],\beta+\gamma},\ E_{S+\chi[\gamma]}) \;\leq\; \max_{1\leq j<\mathtt{N}} \Big|\; \Big(\; A^{\sigma,\,\sigma}_{K+\Delta[\gamma],\beta+\gamma}\;\Big)^{(\pm j)} \;\Big|_{\rho-\zeta}$$

From here and using (139) we obtain estimate (124) in the statement of Lemma 10.

To estimate $\mu^{S+\chi[\gamma],\sigma}_{K+\Delta[\gamma],\beta+\gamma}$, $\sigma\in\{s,u\}$, we use a Neumann series as in the proof of Proposition 2 to obtain

$$\Pi^\sigma_{S+\chi[\gamma]} = \Pi^\sigma_S - \Pi^\sigma_S \begin{pmatrix} 0 & Z^\sigma_\chi[\gamma] \\ \chi^\sigma[\gamma] & 0 \end{pmatrix} \Pi^\sigma_S + \mathcal{E}_{\Pi^\sigma_{S+\chi[\gamma]}}$$

where $\mathcal{E}_{\Pi^\sigma_{S+\chi[\gamma]}}$ is of higher order and, for $\sigma,\ \sigma',\ \sigma''$ a permutation of the letters $\{s,u,c\}$, we have denoted

$$Z^\sigma_\chi \;:=\; \chi^{\sigma',\sigma}\,\Pi^{\sigma'}_0 \begin{pmatrix} \mathrm{Id}_{E^{\sigma'}_0} & S^{\sigma'',\sigma} \\ S^{\sigma',\sigma''} & \mathrm{Id}_{E^{\sigma''}_0} \end{pmatrix}^{-1} + \chi^{\sigma'',\sigma}\,\Pi^{\sigma''}_0 \begin{pmatrix} \mathrm{Id}_{E^{\sigma'}_0} & S^{\sigma'',\sigma'} \\ S^{\sigma',\sigma''} & \mathrm{Id}_{E^{\sigma''}_0} \end{pmatrix}^{-1}$$

$$+\Big(S^{\sigma',\sigma}\Pi^{\sigma'}_0 + S^{\sigma'',\sigma}\Pi^{\sigma''}_0\Big) \begin{pmatrix} \mathrm{Id}_{E^{\sigma'}_0} & S^{\sigma'',\sigma} \\ S^{\sigma,\sigma''} & \mathrm{Id}_{E^{\sigma''}_0} \end{pmatrix}^{-1} \begin{pmatrix} 0 & \chi^{\sigma'',\sigma'} \\ \chi^{\sigma',\sigma''} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id}_{E^{\sigma'}_0} & S^{\sigma'',\sigma'} \\ S^{\sigma',\sigma''} & \mathrm{Id}_{E^{\sigma''}_0} \end{pmatrix}^{-1}$$

Note that, for $1\leq j\leq \mathtt{N}$,

$$\begin{aligned}
\Big|\; \Big(\; A^{\sigma,\,\sigma}_{K+\Delta[\gamma],\beta+\gamma}[S]\;\Big)^{(\pm j)} \;\Big|_{\rho-\zeta} &= \Big|\; \Big(\; \Pi^\sigma_{S+\chi[\gamma]}\circ T_\omega \;\; A_{K+\Delta[\gamma],\beta+\gamma} \;\; \Pi^\sigma_{S+\chi[\gamma]}\;\Big)^{(\pm n)} \;\Big|_{\rho-\zeta} \\
&\leq \Big|\; \Big(\; A^{\sigma,\sigma}_{K,\beta}[S] \;+\; \mathcal{E}_{\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma}}\;\Big)^{(\pm j)} \;\Big|_{\rho-\zeta} \\
&\leq \Big|\; \Big(\; A^{\sigma,\sigma}_{K,\beta}[S]\;\Big)^{(\pm j)} \;\Big|_{\rho-\zeta} + \Big|\; \widetilde{\mathcal{E}}_{\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} \;\Big|_{\rho-\zeta}
\end{aligned}$$

where

$$\mathcal{E}_{\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} := A^{\sigma,\ \sigma}_{K+\Delta[\gamma],\beta+\gamma}[S] - A^{\sigma,\sigma}_{K,\beta}[S],$$

$$\widetilde{\mathcal{E}}_{\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} := \left(A^{\sigma,\sigma}_{K,\beta}[S] \ + \ \mathcal{E}_{\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} \right)^{(\pm j)} \ - \ \left(A^{\sigma,\sigma}_{K,\beta}[S] \right)^{(\pm j)}$$

and, as before, we can estimate

$$\begin{aligned} \left| \ \widetilde{\mathcal{E}}_{\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} \ \right|_{\rho-\zeta} &\leq C \ (\ C_h(A_{K,\beta}, \ E_S) \ )^{j-1} \ \left| \ \mathcal{E}_{\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} \ \right|_{\rho-\zeta} \\ &\prec \frac{C \ M}{\zeta^{2\tau}} \left(\ | \ e_{\mathscr{R}} \ |_\rho \ + \ | \ e_{\mathscr{S}} \ |_\rho \ + \ | \ e \ |_\rho \right) \end{aligned} \tag{141}$$

Hence, we can take

$$\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma} \ \leq \ \mu^{S,\sigma}_{K,\beta} + \left| \ \widetilde{\mathcal{E}}_{\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} \ \right|_{\rho-\zeta} ,$$

$$C_h(A_{K+\Delta[\gamma],\beta+\gamma}, \ E_{S+\chi[\gamma]}) \ \leq \ \max_{1\leq j<\mathtt{N}} \left| \ \left(\ A^{\sigma,\ \sigma}_{K+\Delta[\gamma],\beta+\gamma}[S] \ \right)^{(\pm j)} \ \right|_{\rho-\zeta}$$

From which, using the estimate

$$\left| \ \mathcal{E}_{\mu^{S,\sigma}_{K+\Delta[\gamma],\beta+\gamma}} \ \right|_{\rho-\zeta} \prec \frac{C \ M}{\zeta^{2\tau}} \left(\ | \ e_{\mathscr{R}} \ |_\rho \ + \ | \ e_{\mathscr{S}} \ |_\rho \ + \ | \ e \ |_\rho \right)$$

estimate (124) follows.

□

CHAPTER VII

CONVERGENCE OF THE QUASI-NEWTON ITERATION.

In this Chapter we prove that the Nash-Moser iteration outlined in Algorithm 1 indeed converges to solutions of the invariance and reducibility equations introduced in Section 4.1.

Throughout this Chapter we use the subscript $\mathtt{k} \in \mathbb{N}$ to denote the state of the variables and parameters of the iteration at step $\mathtt{k}$ of Algorithm 1, that is:

- We choose the domain losses to be:

$$\zeta_{\mathtt{k}} \ := \ \frac{\rho_0}{2^{\mathtt{k}+2}}$$

  and note that $\rho_{\mathtt{k}} \ := \rho_0 - \sum_{0 \leq j \leq \mathtt{k}} \zeta_j > \rho_0/2$.

- Given $K_{\mathtt{k}} \in \mathcal{A}_{\rho_{\mathtt{k}}}^{\mathbb{C}^n}$, $\beta_{\mathtt{k}} \in \mathbb{C}^c$, $S_{\mathtt{k}}^{\sigma} \in \mathcal{A}_{\rho_{\mathtt{k}}}^{\mathcal{L}(E_0^{\sigma}, E_0^{\sigma'} \oplus \sigma'')}$, $\sigma \in \{s, u, c\}$ and $W_{\mathtt{k}} \in \mathcal{A}_{\rho_{\mathtt{k}}}^{GL(\mathbb{C}^c)}$ such that

$$\begin{aligned} K_{\mathtt{k}} \left(\ \mathbb{T}_{\rho_{\mathtt{k}}}^d \ \right) \ &\subset \mathcal{D}_{\rho_0, 3\eta_0} \left(K_0, \ \beta_0 \right), \\ E_{S_{\mathtt{k}}}^{\sigma} \ &\in \ \mathcal{V}_{\eta_0^{\mathscr{S}}} \left(E_0^{\sigma} \ \right) \ , \quad \sigma \in \{s, u, c\} \end{aligned} \tag{142}$$

  denote the errors of the approximate solutions of the invariance equation at step $\mathtt{k}$ by:

$$\begin{aligned} e_{\mathtt{k}} \ &:= \ \mathscr{I} \left[K_{\mathtt{k}}, \ \beta_{\mathtt{k}} \right], \\ e_{\mathscr{S}, \mathtt{k}}^{\sigma} \ &:= \ \mathscr{S}^{\sigma} \left[K_{\mathtt{k}}, \ \beta_{\mathtt{k}}, \ S_{\mathtt{k}}^{\sigma, \sigma'}, \ S_{\mathtt{k}}^{\sigma, \sigma''} \right], \\ e_{\mathscr{R}, \mathtt{k}} \ &:= \ \mathscr{R} \left[K_{\mathtt{k}}, \ \beta_{\mathtt{k}}, \ S_{\mathtt{k}}, \ U_{\mathtt{k}} \right] \end{aligned}$$

Lemma 11 (Recursive Lemma)**.** *Assume that the splitting* $E_{S_{\mathtt{k}}}$ *is* $C_{h,\mathtt{k}},\ \mu^{S_{\mathtt{k}},s}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{S_{\mathtt{k}},u}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{S_{\mathtt{k}},c,+}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{S_{\mathtt{k}},c,-}_{K_{\mathtt{k}},\beta_{\mathtt{k}}}$ *- whiskered with respect to the cocycle* $A_{K_{\mathtt{k}},\beta_{\mathtt{k}}}$*, and that the splitting* E_0 *is* $C_{h,\mathtt{k}},\ \mu^{0,s}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{0,u}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},$ $\mu^{0,c,+}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{0,c,-}_{K_{\mathtt{k}},\beta_{\mathtt{k}}}$*-whiskered with respect to the cocycle* $A_{K_{\mathtt{k}},\beta_{\mathtt{k}}}$*. Let* $\xi \in \mathbb{C}^c$*,* $\omega \in \mathbb{T}^d$ *and let* $\Lambda \in \mathbb{C}^{c\times c}$*,* $\Lambda = diag[\lambda_1,\ \ldots,\ \lambda_c]$ *be such that*

$$\omega \in DC(\nu,\tau)$$

$$Spec(\Lambda) \in DC^{1st}_{\omega}(\nu,\tau)\ \cap DC^{2nd}_{\omega}(\nu,\tau)$$

Assume furthermore that the following non-degeneracy condition is satisfied:

$$M_{\mathtt{k}} := \left(\min_{1\le j\le c} \left| \left[\mathtt{Lin}\left(U^{(-1)}_{\mathtt{k}} \circ T_\omega\ \eta^{\daleth}_{\mathscr{R},\mathtt{k}} \right) \right]^{\widehat{}}_{jj} (0) \right| \right)^{-1} \ <\ \infty$$

where the expression for $\eta^{\daleth}_{\mathscr{R},\mathtt{k}}$ *is given in Lemma 9.*

Define the corrections $\gamma_{\mathtt{k}} \in \mathbb{C}^c$*,* $\Delta_{\mathtt{k}}[\gamma_{\mathtt{k}}]$*,* $K_{\mathtt{k}+1} \in \mathcal{A}^{\mathbb{C}^n}_{\rho_{\mathtt{k}+1}}$*,* $\chi^{\sigma}_{\mathtt{k}}[\gamma_{\mathtt{k}}]$*,* $S^{\sigma}_{\mathtt{k}+1} \in \mathcal{A}^{\mathcal{L}(E^{\sigma}_0, E^{\sigma'\oplus\sigma''}_0)}_{\rho_{\mathtt{k}+1}}$*,* $W_{\mathtt{k}+1} \in \mathcal{A}^{GL(\mathbb{C}^c)}_{\rho_{\mathtt{k}+1}}$ *as follows:*

$$\Delta_{\mathtt{k}}\ :=\ \Gamma_{\mathscr{I}}(-e_{\mathtt{k}})\ ,$$

$$\chi^{\sigma,\sigma'}_{\mathtt{k}}\ :=\ \Gamma_{\mathscr{S}^{\sigma,\sigma'}}(-e^{\sigma,\sigma'}_{\mathscr{S},\mathtt{k}};\ \Delta_{\mathtt{k}}),\ \sigma \neq \sigma'\ ,\quad \chi_{\mathtt{k}}\ :=\ \prod_{\sigma\neq\sigma'} \chi^{\sigma,\sigma'}_{\mathtt{k}}$$

$$(W,\ \gamma\ )\ :=\ (\ \Gamma_{\mathscr{R},\ \xi}(e_{\mathscr{R}}),\ \gamma_{\mathscr{R}}(e_{\mathscr{R}});\ \Delta,\ \chi\ )$$

Denote:

$$H_{\mathtt{k}} := \max_{\sigma\in\{s,u,c\}} \Big\{ \ |\ \Delta_{\mathtt{k}}[\gamma_{\mathtt{k}}]\ |_{\rho_{\mathtt{k}+1}}\ ,\ |\ \chi_{\mathtt{k}}[\gamma_{\mathtt{k}}]\ |_{\rho_{\mathtt{k}+1}}\ ,\ |\ W_{\mathtt{k}}\ |_{\rho_{\mathtt{k}+1}}\ ,\ |\ \gamma_{\mathtt{k}}\ |\ ,$$
$$\mu^{S_{\mathtt{k}}+\chi_{\mathtt{k}}[\gamma_{\mathtt{k}}],\sigma}_{K_{\mathtt{k}}+\Delta_{\mathtt{k}}[\gamma_{\mathtt{k}}],\beta_{\mathtt{k}}+\gamma_{\mathtt{k}}} - \mu^{S_{\mathtt{k}},\sigma}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{0,\sigma}_{K_{\mathtt{k}}+\Delta_{\mathtt{k}}[\gamma_{\mathtt{k}}],\beta_{\mathtt{k}}+\gamma_{\mathtt{k}}} - \mu^{0,\sigma}_{K_{\mathtt{k}},\beta_{\mathtt{k}}} \Big\}$$

$$L_{\mathtt{k}} := \max \Big\{ \ |\ K_{\mathtt{k}}\ |_{\rho_{\mathtt{k}}}\ ,\ |\ f\ |_{\rho_{\mathtt{k}}}\ ,\ |\ S_{\mathtt{k}}\ |_{\rho_{\mathtt{k}}}\ ,\ |\ U_{\mathtt{k}}\ |_{\rho_{\mathtt{k}}}\ ,\ \left|\ U^{(-1)}_{\mathtt{k}}\ \right|_{\rho_{\mathtt{k}}}\ ,$$
$$C_h(A_{K+\Delta[\gamma],\beta+\gamma},\ E_{S+\chi[\gamma]}) - C_h(A_{K,\beta},\ E_S)\ \Big\}$$

$$\epsilon_{\mathrm{k}} := \max\left\{ \mid e_{\mathrm{k}} \mid_{\rho_{\mathrm{k}}},\ \max_{\sigma\in\{s,u,c\}}\left\{ \mid e^{\sigma}_{\mathscr{S},\mathrm{k}} \mid_{\rho_{\mathrm{k}}} \right\},\ \mid e_{\mathscr{R},\mathrm{k}} \mid_{\rho_{\mathrm{k}}} \right\}$$

If there exist positive constants C_{M_0}, C_{L_0}, which depend only on M_0, L_0, such that

$$\begin{cases} M_{\mathrm{k}} \ \leq\ C_{M_0} \\ \\ L_{\mathrm{k}} \ \leq\ C_{L_0} \end{cases} \tag{143}$$

then

$$H_{\mathrm{k}} \ \leq\ \frac{3\ C_{M_0}\ C_{L_0}}{\rho_0^{2\tau}}\ 2^{2\tau(\mathrm{k}+2)}\ \epsilon_{\mathrm{k}} \tag{144}$$

If, furthermore,

$$\begin{aligned} K_{\mathrm{k}+1}\left(\ \mathbb{T}^d_{\rho_{\mathrm{k}+1}}\ \right)\ &\subset \mathcal{D}_{\rho_0,2\eta_0}\left(K_0,\ \beta_0\right), \\ E^{\sigma}_{S_{\mathrm{k}+1}}\ &\in\ \mathcal{V}_{\eta_0^{\mathscr{S}}}\left(E^{\sigma}_0\ \right)\ ,\quad \sigma\in\{s,u,c\} \end{aligned} \tag{145}$$

then:

$$\epsilon_{\mathrm{k}+1} \leq \frac{9\ C^2_{M_0}\ C_{L_0}}{\rho_0^{4\tau}}\ 2^{4\tau(\mathrm{k}+2)}\ \epsilon^2_{\mathrm{k}} \tag{146}$$

Proof. Using estimates (115), (116), (117), (118), and (124) in Lemma 10 we obtain

$$H_{\mathrm{k}}\ \leq\ \frac{M_{\mathrm{k}}}{\zeta_{\mathrm{k}}^{2\tau}}\ 3\epsilon_{\mathrm{k}}\ L^{25}_{\mathrm{k}}$$

and from here we obtain estimate (144). If condition (145) is satisfied, we can apply estimates (120), (122), (123) in Lemma 10 to obtain

$$\epsilon_{\mathrm{k}+1}\ \leq\ \frac{M^2_{\mathrm{k}}}{\zeta_{\mathrm{k}}^{4\tau}}\ (3\epsilon_{\mathrm{k}})^2\ L^{96}_{\mathrm{k}}$$

and from here we obtain estimate (146). □

7.1 Proof of Theorem 1

Let

$$C_* \ := \ \frac{9\ C_{M_0}^2\ C_{L_0}}{\rho_0^{4\tau}}$$

and let us assume, without loss of generality, that $C_* \geq 1$. Let

$$\epsilon_* \ := \ (2^{2\tau} C_*)^{-1}$$

Given $0 \ < \ \bar{\epsilon}_0 \ < \ \epsilon_*$, we define a sequence $\{\bar{\epsilon}_{\mathtt{k}}\}_{\mathtt{k}\in\mathbb{N}}$ recursively by:

$$\bar{\epsilon}_{\mathtt{k}+1} \ := \ C_*\ 2^{4\tau(\mathtt{k}+2)}\ \bar{\epsilon}_{\mathtt{k}}{}^2$$

Note that

$$\begin{aligned} \bar{\epsilon}_{\mathtt{k}} \ &= \ C_*^{\sum_{j=0}^{\mathtt{k}-1} 2^j} \left(2^{2\tau}\right)^{\sum_{j=1}^{\mathtt{k}-1} (\mathtt{k}-j)\ 2^{j-1}}\ \bar{\epsilon}_0^{2^{\mathtt{k}}} \\ &\leq \left(\ C_*\ 2^{2\tau}\ \bar{\epsilon}_0\right)^{2^{\mathtt{k}}} \end{aligned} \tag{147}$$

where we have used the estimates

$$\sum_{j=0}^{\mathtt{k}-1} 2^j = 2^{\mathtt{k}} - 1\ , \qquad \sum_{j=1}^{\mathtt{k}-1} (\mathtt{k}-j)\ 2^{j-1} \ \leq \ (\mathtt{k}+1)2^{\mathtt{k}}$$

Hence, since $C_* 2^{2\tau}\ \bar{\epsilon}_0 \ < \ 1$, the series

$$r(\bar{\epsilon}_0) \ := \ \frac{3\ C_{M_0}\ C_{L_0}}{\rho_0^{2\tau}} \ \sum 2^{2\tau(\mathtt{k}+2)}\ \bar{\epsilon}_{\mathtt{k}}$$

is absolutely convergent.

We now state the condition numbers which are used to ensure that the following geometric conditions are satisfied throughout the iterative procedure:

1. $K_{\mathtt{k}}(\mathbb{T}^d_\rho) \subset \mathcal{D}_{\rho_0, 2\eta_0}(K_0,\ \beta_0)$,

2. $E^\sigma_{S_{\mathtt{k}}} \ \in \ \mathcal{V}_{\eta_0^{\mathscr{S}}}(E^\sigma_0\ )\ , \ \ \sigma \in \{s, u, c\}$,

3. $A_{K_{\mathtt{k}},\beta_{\mathtt{k}}} \in \mathcal{A}_{\rho_{\mathtt{k}}}^{GL(\mathbb{C}^n)\times\mathbb{T}^d_\rho}$.

4. The splitting $E_{S_{\mathtt{k}}}$ is $C_{h,\mathtt{k}},\ \mu^{S_{\mathtt{k}},s}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{S_{\mathtt{k}},u}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{S_{\mathtt{k}},c,+}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{S_{\mathtt{k}},c,-}_{K_{\mathtt{k}},\beta_{\mathtt{k}}}$ - whiskered with respect to the cocycle $A_{K_{\mathtt{k}},\beta_{\mathtt{k}}}$, and the splitting E_0 is $C_{h,\mathtt{k}},\ \mu^{0,s}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{0,u}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{0,c,+}_{K_{\mathtt{k}},\beta_{\mathtt{k}}},\ \mu^{0,c,-}_{K_{\mathtt{k}},\beta_{\mathtt{k}}}$- whiskered with respect to the cocycle $A_{K_{\mathtt{k}},\beta_{\mathtt{k}}}$.

5. The non-degeneracy condition $M_{\mathtt{k}} \le C_{M_0}$.

By taking ϵ_0 sufficiently small so that $r(\epsilon_0)$ is smaller than a finite set of positive condition numbers, we will ensure that the geometric conditions (1) - (5) above are automatically satisfied throughout the iterative procedure.

7.1.1 Condition number for geometric condition (1)

We require

$$r(\epsilon_0) < \eta_0 \tag{148}$$

Then, using induction on $\mathtt{k} \in \mathbb{N}$ and Lemma 11, we have

$$\begin{aligned} K_{\mathtt{k}}\left(\ \mathbb{T}^d_{\rho_{\mathtt{k}}}\ \right) &\subset \mathcal{D}_{\rho_0,\eta_0+\sum_{j=0}^{\mathtt{k}} H_j}\left(K_0,\ \beta_0\right), \\ &\subset \mathcal{D}_{\rho_0,\eta_0+r(\epsilon_0)}\left(K_0,\ \beta_0\right), \\ &\subset \mathcal{D}_{\rho_0,2\eta_0}\left(K_0,\ \beta_0\right) \end{aligned} \tag{149}$$

7.1.2 Condition number for geometric condition (2)

Besides condition (148), we require

$$r(\epsilon_0) < \eta_0^{\mathscr{S}} \tag{150}$$

Then, using induction on $\mathtt{k} \in \mathbb{N}$ and Lemma 11, we have

$$
\begin{aligned}
E^{\sigma}_{S_{\mathtt{k}}} \;\in\; & \mathcal{U}^{\sum_{j=0}^{\mathtt{k}} H_j}_{\mathcal{V}^{\sigma}_{\rho}}\left(E^{\sigma}_{0}\right), \\
& \subset \mathcal{U}^{r(\epsilon_0)}_{\mathcal{V}^{\sigma}_{\rho}}\left(E^{\sigma}_{0}\right), \\
& \subset \mathcal{U}^{\eta_0^{\mathscr{S}}}_{\mathcal{V}^{\sigma}_{\rho}}\left(E^{\sigma}_{0}\right), \\
& \subset \mathcal{V}_{\eta_0^{\mathscr{S}}}\left(E^{\sigma}_{0}\right)
\end{aligned}
\tag{151}
$$

7.1.3 Condition number for geometric condition (3)

Let

$$
\eta_0^{GL}\left(A_{K_0,\beta_0},\ \mathcal{A}^{GL(\mathbb{C}^n\times\mathbb{T}^d_{\rho_0})}_{\rho_0,\ \omega}\right) \;>\; 0
$$

be such that, if $B \in \mathcal{A}^{\mathcal{L}(\mathbb{C}^n\times\mathbb{T}^d_{\rho_0})}_{\rho_0,\ \omega}$ satisfies

$$
|\ B - A_{K_0,\beta_0}\ |_{\rho_0} \;<\; \eta_0^{GL}\left(A_{K_0,\beta_0},\ \mathcal{A}^{GL(\mathbb{C}^n\times\mathbb{T}^d_{\rho_0})}_{\rho_0,\ \omega}\right)
$$

then $B \in \mathcal{A}^{GL(\mathbb{C}^n\times\mathbb{T}^d_{\rho_0})}_{\rho_0,\ \omega}$.

Besides conditions (148), (150), we require

$$
r(\epsilon_0) < \eta_0^{GL}\left(A_{K_0,\beta_0},\ \mathcal{A}^{GL(\mathbb{C}^n\times\mathbb{T}^d_{\rho_0})}_{\rho_0,\ \omega}\right)
\tag{152}
$$

Then, using induction on $\mathtt{k} \in \mathbb{N}$ and Lemma 11, we have

$$
\begin{aligned}
|\ A_{K_{\mathtt{k}},\beta_{\mathtt{k}}} - A_{K_0,\beta_0}\ |_{\rho_{\mathtt{k}}} \;\le\; & \sum_{j=0}^{k} H_j \\
\le\; & r(\epsilon_0), \\
\le\; & \eta_0^{GL}\left(A_{K_0,\beta_0},\ \mathcal{A}^{GL(\mathbb{C}^n\times\mathbb{T}^d_{\rho_0})}_{\rho_0,\ \omega}\right)
\end{aligned}
\tag{153}
$$

and so, $A_{K_{\mathtt{k}},\beta_{\mathtt{k}}} \;\in\; \mathcal{A}^{GL(\mathbb{C}^n\times\mathbb{T}^d_{\rho_0})}_{\rho_0,\ \omega}$.

7.1.4 Condition number for geometric condition (4)

Let

$$\eta_0^{Hyp} \ := \ \min_{\sigma \in \{s,u\}} \left\{ \ 1 - \mu^{\sigma}_{K_0,\beta_0} \ , \quad 1 - \mu^{\sigma}_{K_0,\beta_0} \ \mu^{c,+}_{K_0,\beta_0}, \ 1 - \mu^{\sigma}_{K_0,\beta_0} \ \mu^{c,-}_{K_0,\beta_0} \ \right\}$$

If, besides conditions (148), (150), (152), we require

$$r(\epsilon_0) \ < \ \eta_0^{Hyp} \tag{154}$$

Then, geometric condition (4) will automatically be satisfied throughout the iterative procedure.

7.1.5 Condition number for geometric condition (5)

Let

$$\eta_0^{M} \ := \ \frac{1}{2} \ M_0$$

Besides conditions (148), (150), (152), (154), we require

$$r(\epsilon_0) \ < \ \eta_0^{Hyp} \tag{155}$$

Given $1 \leq j \leq c$, denote

$$a_{\mathtt{k}}(j) \ := \ \left[\ \mathtt{Lin} \left(\ U_{\mathtt{k}}^{(-1)} \circ T_\omega \ \eta^{\daleth}_{\mathscr{R},\mathtt{k}} \ \right) \ \right]^{\widehat{}}_{jj} (0)$$

Then,

$$| \ a_{\mathtt{k}}(j) - a_0(j) \ | \ \leq \ \sum_{l=0}^{\mathtt{k}} | \ a_l(j) - a_{l-1}(j) \ | \ \leq \ \sum_{l=0}^{\mathtt{k}} H_{\mathtt{k}} \ \leq \ r(\epsilon_0) \ \leq \frac{M_0}{2}$$

and so $M_{\mathtt{k}} \leq 2M_0$.

$\square$

Part II: Regularity properties of the boundary of a domain that imply it is compensated

CHAPTER VIII

INTRODUCTION

Let $\mathcal{B}$ be a Banach space. A *Domain* in $\mathcal{B}$ is a connected open set of $\mathcal{B}$. Given a domain $\mathcal{D}$ and a differentiable function $f : \mathcal{D} \mapsto \mathbb{R}$, the mean value theorem shows that for any points x, $y \in D$ and any C^1 path γ contained in $\mathcal{D}$ joining x and y we have

$$|f(x) - f(y)| = \left| \int_0^1 Df(\gamma(t))\gamma'(t)dt \right| \leq \sup_{z \in D} |f(z)| \int_0^1 |\gamma'(t)|\, dt \tag{156}$$

The inequality (156) does not allow us to conclude that f is Lipschitz when it is C^1. An important ingredient to reach this conclusion is that the geometry of $\mathcal{D}$ has to be such that the *distance on $\mathcal{D}$ inherited from $\mathcal{B}$* satisfies:

$$d_{\mathcal{D}}(x, y) := \inf_{\gamma \text{ joins } x,y} \text{length}(\gamma) \;\; \leq \;\; C\,|x - y| \qquad \forall\, x, y \in \mathcal{D}. \tag{157}$$

Following [40], we make the following definition:

Definition 1. *When a domain $\mathcal{D}$ satisfies property* (157)*, we will say that $\mathcal{D}$ is* **compensated**.

The notion of compensation makes sense in the generality of Banach spaces and (156) remains true when Df is understood as a weak (Gateaux) derivative. Hence, the mean value theorem shows that in compensated domains functions with bounded Gateaux derivatives are Lipschitz. Easy examples show that there are non-compensated domains for which non-Lipschitz functions with bounded Gateaux derivatives exist.

In [11] it was shown that the property of compensation is crucial for many other properties in function theory, including Hadamard-Kolmogorov interpolation, existence of smoothing operators, etc. In [40], it was shown that compensation also plays a role in convergence properties of Whitney differentiable functions, which in turn are important for KAM theory and renormalization. The values of the constants in (157) affect many other inequalities.

This paper contains the following results: In section 2, we present Theorem 2, which shows that local regularity properties (stated in that section and denoted $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$) of the boundary of $\mathcal{D} \in \mathbb{R}^n$ imply that $\mathcal{D}$ is compensated. In section 3, we present two counterexamples, in finite dimensions, for tentative generalisations of the theorem. In section 4, we present a counterexample that shows that Theorem 2 cannot be generalised to the case where $\mathcal{D}$ is a domain in an infinite dimensional Banach space, even a Hilbert space. Hence, an open problem is to find local regularity properties of domains in infinite dimensions which guarantee compensation.

CHAPTER IX

CONDITIONS IMPLYING COMPENSATION IN FINITE DIMENSIONS.

In this section we present regularity conditions on the boundary of a domain $\mathcal{D}$ in $\mathbb{R}^n$ that imply that $\mathcal{D}$ is compensated.

Throughout this section we will consider a domain $\mathcal{D} \in \mathbb{R}^n$ satisfying the following conditions:

$\mathbf{C}_1$: $\mathcal{D}$ is a bounded set of $\mathbb{R}^n$.

$\mathbf{C}_2$: The boundary of $\mathcal{D}$, $\partial\mathcal{D}$, is a path-connected $(n-1)$- dimensional *immersed* C^1 manifold in $\mathbb{R}^n$.

There are noncompensated domains satisfying conditions $\mathbf{C}_1$ and $\mathbf{C}_2$ (cf. Section 3.1, example 1), and we will impose an additional regularity condition on the boundary of $\mathcal{D}$ that, as we will prove in Theorem 2, implies that $\mathcal{D}$ is compensated. Before we can define condition $\mathbf{C}_3$, we need to introduce some objects and notation.

9.1 Notation and preliminaries.

In this Section we introduce some geometric objects and notation that we will use throughout section 2.

Because of $\mathbf{C}_2$, any $x \in \partial\mathcal{D}$ lies in the image of a local embedding

$$\iota_x : \mathbb{R}^{n-1} \supset U \longrightarrow \partial\mathcal{D} \subset \mathbb{R}^n$$

where U is an open set. This allows to define a unit vector normal field on $\partial\mathcal{D}$, pointing outwards of $\mathcal{D}$, which we call $N \subset \partial\mathcal{D} \times \mathbb{R}^n$ as follows: We will say that $v \in \mathbb{R}^n$ is in $T^{\perp}\partial\mathcal{D}(x)$ if $v \in T^{\perp}\iota_x(U)$. We note that the unit normal vector to $\partial\mathcal{D}$ at x so defined depends only on $x \in \partial\mathcal{D}$ and not on the local embedding ι_x. We will denote this vector by $n(x)$, and by $\ell(x)$ the semi-line normal to $\partial\mathcal{D}$ at x, that is, $\ell(x) = \{z \in \mathbb{R}^n : z = x + \lambda n(x), \lambda > 0\}$

We emphasize that, since $\partial\mathcal{D}$ is not necessarily embedded in $\mathbb{R}^n$, N could fail to admit an extension as a continuous vector field in $\mathbb{R}^n$ (see Counterexample 1 in section 3).

Definition 2. *If $X \subset \mathbb{R}^n$ is a set, we will call each connected component of $X \cap \partial\mathcal{D}$ a X* **-slice***.*

If $x \in \partial\mathcal{D} \cap X$, we will denote by $\partial\mathcal{D}_X(x)$ the ***X -slice containing*** *x.*

Definition 3. *Let $X \subset \mathbb{R}^n$ be a set and $x, y \in X \cap \partial\mathcal{D}$. We will denote $\ell_X(x) = \ell(x) \cap X$. We also make the following definitions:*

$$h(x) = \inf\{|z - x| : \quad z \in \partial\mathcal{D} \cap \ell(x)\}$$

$$H(x) = z \in \ell(x), \quad \textit{where} \;\; h(x) = |z - x|$$

$$h_y(x, X) = \inf\{|z - x| : \quad z \in \partial\mathcal{D}_X(y) \cap \ell(x)\}$$

$$H_y(x, X) = z \in \ell(x), \quad \textit{where} \;\; h_y(x, X) = |z - x|$$

Definition 4. *Let $x, v \in \mathbb{R}^n$ and $\epsilon \in [0, \pi/2] \subset \mathbb{R}$. We define the* ***cone of aperture ϵ, of vertex*** *x* ***and of axis*** *v as the set:*

$$C_\epsilon(x, v) = \left\{ x + w \in \mathbb{R}^n : \frac{| < v, w > |^2}{\|v\|^2 \|w\|^2} > 1 - \sin^2(\epsilon) \right\}$$

This cone is composed of two components whose closures intersect at the vertex x. We denote these components by $C_\epsilon^\pm(x,v)$.

Clearly, $C_\epsilon(x,v) = C_\epsilon(x,-v)$ and $C_\epsilon^+(x,v) = C_\epsilon^-(x,-v)$

Definition 5. *Given $\epsilon > 0$, we will say that an open ball B (in $\mathbb{R}^n$) is an ϵ-* **uniform ball for** ∂D *if, for any $x, y \in \partial\mathcal{D} \cap B$, it holds that*

$$| < n(x), n(y) > |^2 > 1 - \sin^2(\epsilon)$$

where $<\ ,\ >$ is the Euclidean inner product in $\mathbb{R}^n$.

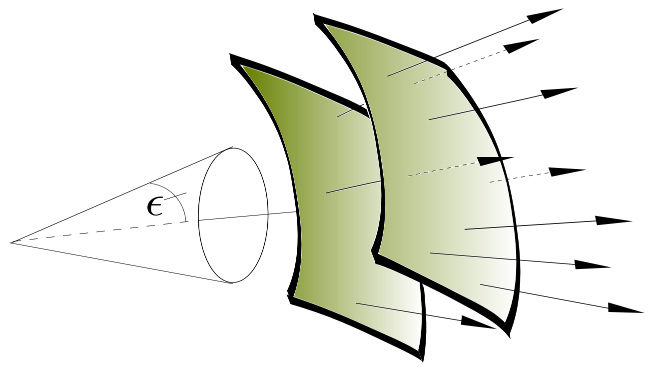

Figure 3: Two non- opposing slices inside an ϵ -uniform ball.

If B is an ϵ-uniform ball for $\partial\mathcal{D}$, then the set of $\mathbb{R}^n$:

$$N_{\partial\mathcal{D}\cap B} = \{n(x) : x \in \partial\mathcal{D} \cap B\}$$

is contained in $C_\epsilon(0,\eta)$, where η is a fixed vector (see Figure 1). We remark that the cone $\mathcal{C}_\epsilon(0,\eta)$ has two opposing sub cones, and normal vectors in an ϵ - uniform ball may lie in these opposing sub cones. We can now state condition $\mathbf{C}_3$:

$\mathbf{C}_3$: Given any $\epsilon > 0$, and $x \in \partial\mathcal{D}$, there exists an ϵ- uniform ball of $\partial\mathcal{D}$ containing x.

Definition 6. *A domain $\mathcal{D}$ that satisfies conditions $\boldsymbol{C}_1$, $\boldsymbol{C}_2$ and $\boldsymbol{C}_3$ is called a* **uniformly planar domain.**

9.2 Uniformly planar domains in $\mathbb{R}^n$ are compensated.

We now state the main theorem of this section.

Theorem 2. *Let $\mathcal{D} \subset \mathbb{R}^n$ satisfy $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, $\boldsymbol{C}_3$. Then $\mathcal{D}$ is compensated.*

Outline of the proof of Theorem 2: We first show, in Lemma 13, that if $\mathcal{D} \subset \mathbb{R}^n$ is uniformly planar then $\partial\mathcal{D}$ is locally connected (with respect to the topology of $\mathbb{R}^n$). Then we prove, in Proposition 14, that if $\partial\mathcal{D}$ is uniformly planar and locally connected with respect to the topology of $\mathbb{R}^n$, then $\mathcal{D}$ is compensated. $\square$

Proposition 3. *Let $B_{\pi/4}$ be a $\pi/4$-uniform ball for $\partial\mathcal{D}$ with diam$(B_{\pi/4}) \leq d$ and such that $\partial\mathcal{D} \cap B_{\pi/4}$ is path connected. Then, for any $x, y \in \partial\mathcal{D} \cap B_{\pi/4}$, $d_{\mathcal{D}}(x, y) \leq 2d$.*

Proof: Let $\Pi_{x,y}$ be the plane that contains $\ell(x)$, and the point y. Because $B_{\pi/4}$ is a $\pi/4$-uniform ball and $\Pi_{x,y}$ contains $n(x)$, $\Pi_{x,y}$ intersects $\partial\mathcal{D} \cap B_{\pi/4}$ transversally. Thus, $\Pi_{x,y} \cap (\partial\mathcal{D} \cap B_{\pi/4})$ is the image of a C^1 curve $\tilde{\gamma}(t)$ such that $\tilde{\gamma}(0) = x, \tilde{\gamma}(1) = y$. Let $\gamma(t) : [0, 1] \to \partial D$ be defined by $\gamma(t) = \tilde{\gamma}(t),\ t \in [0, 1]$.

We can see that length$(\gamma) \leq 2d$ as follows: Let $S_d(\gamma(0), \gamma'(0)) \subset \Pi_{x,y}$ be a square with one vertex at $\gamma(0)$ and diagonal parallel to $\gamma'(0)$, and such that the length of the sides of $S_d(\gamma(0), \gamma'(0))$ is d. We note that $y \in S_d(\gamma(0), \gamma'(0))$. As $B_{\pi/4}$ is a $\pi/4$-uniform ball and γ is contained in the plane $\Pi_{x,y}$, the set

$$\{\gamma'(s)\}_{s\in[0,1]} \subset T(\Pi_{x,y}) = \Pi_{x,y}$$

is contained in the cone $C^+_{\pi/4}(\gamma(0), \gamma'(0))$. The claim now follows from the following elementary geometric argument: Label the two sides of the square $S_d(\gamma(0), \gamma'(0))$ which contain the vertex $x = \gamma(0)$ by $\hat{u}_1$, $\hat{u}_2$. Given $z \in S_d(\gamma(0), \gamma'(0))$, let $\Pi_i(z)$ be the orthogonal projection of $S_d(\gamma(0), \gamma'(0))$ onto $\hat{u}_i$, $i = 1, 2$. Let

$$\gamma(t) = (\Pi_1(\gamma(t)), \Pi_2(\gamma(t))) := (u_1(t), u_2(t))$$

be a parameterization of γ in these coordinates. As

$$\{\gamma'(s)\}_{s\in[0,1]} \subset C^{+}_{\pi/4}(\gamma(0), \gamma'(0))$$

we have that $u_1'(t) \geq 0$ and $u_2'(t) \geq 0$, for $t \in [0,1]$. Thus,

$$d_{\mathcal{D}}(x,y) = \int_0^1 \sqrt{u_1'^2(t) + u_2'^2(t)}\ dt \leq \int_0^1 u_1'(t) + u_2'(t)\ dt \leq 2d \ \ \square$$

Lemma 12. ***(Ordering):*** *Let $B_{\pi/8}$ be a $\pi/8$- uniform ball, $x \in B_{\pi/8}$ and $y \in \partial\mathcal{D}_{B_{\pi/8}}(x)$. If $h(x) > 0$, then $h(y) > 0$.*

Proof: Since $h(x) > 0$, we have that $H(x) \neq x$. Let us assume, without loss of generality, that $H(x) \in B_{\pi/8}$. Let Π be the plane containing $n(x)$ and y. $\partial\mathcal{D}_{B_{\pi/8}}(H(x))$ intersects $n(y)$ at a point w. If $h(y) = 0$, there exists $z \in \partial\mathcal{D}$ contained in the interior of the segment

$$[y, w] \subset n(y)$$

We now observe that $\partial\mathcal{D}_{B_{\pi/8}}(z) \cap \Pi$ is a curve bounded between the curves

$$\partial\mathcal{D}_{B_{\pi/8}}(H(x)) \cap \Pi \ \ , \ \partial\mathcal{D}_{B_{\pi/8}}(y) \cap \Pi \ \ , \ n(x)$$

Hence, it intersects $n(x)$ in the interior of the segment $[x, H(x)]$, but this contradicts the definition of $H(x)$. $\square$

Remark 24. *In Lemma 12 it can be shown that $H(y) \in \partial_{\mathcal{D}_{\pi/8}}(H(x))$, but we will not need this fact. Graphically, Lemma 12 means that, inside a $\pi/8$- uniform ball, all slices can be totally ordered by their height with respect to a fixed slice.*

Given $\epsilon > 0$, and $x \in \partial\mathcal{D}$, we denote by $B_{\pi/8}(x,\epsilon)$ a $\pi/8$-uniform ball of radius ϵ centered at x. Such a uniform ball exists because $\mathcal{D}$ is uniformly planar. Consider the set:

$$\mathfrak{A} = \{x \in \partial\mathcal{D} : \forall\epsilon > 0,\ \exists\ x_\epsilon \in B_{\pi/8}(x,\epsilon) \cap \partial\mathcal{D}, \text{such that } x_\epsilon \notin \partial\mathcal{D}_{B_{\pi/8}(x,\epsilon)}(x)\}$$

The set $\mathfrak{A}$ is composed of the points in $\partial\mathcal{D}$ which are accumulation points for different slices of $\partial\mathcal{D} \cap B_{\pi/8}(x,\epsilon)$, for some ϵ. To prove that uniformly planar domains are locally connected (Lemma 13), we will show that $\mathfrak{A}$ is empty. Note that, since $\mathcal{D}$ is uniformly planar, $\mathfrak{A}$ can be equivalently defined as follows: $x_0 \in \mathfrak{A}$ if, and only if, x_0 is an accumulation point of the set

$$\{\overline{\mathcal{D}} \cap \ell(x_0)\} \subset \ell(x_0).$$

Proposition 4. *Let $\mathcal{D}$ be a uniformly planar domain. If there is a path in $\partial\mathcal{D}$ that joins some point $z \in \partial\mathcal{D}$ with $x_0 \in \mathfrak{A}$, then $z \in \mathfrak{A}$.*

Proof: Let γ be a curve with $\gamma(0) = z \in \partial\mathcal{D}$ and $\gamma(1) = x_0 \in \mathfrak{A}$. It is easy to see that $\mathfrak{A}$ is a closed set. Hence, to show that $z \in \mathfrak{A}$, it suffices to show that $\mathfrak{A} \cap \gamma$ is open in the connected set γ. If $\mathfrak{A} \cap \gamma$ were not open in γ, take a point $\gamma(t_0) \in \mathfrak{A} \cap \gamma$ such that there is a sequence of points $\{\gamma(t_k)\}_{k\in\mathbb{N}}$ in γ converging to $\gamma(t_0)$ and which are not in $\mathfrak{A}$. We have: $h(\gamma(t_k)) > 0$, whereas $h(\gamma(t_0)) = 0$.

To illustrate the situation, in Figure 4 we have drawn a plane that contains $n(\gamma(t_0))$ and a point $\gamma(t_k)$. Graphically, the slices that accumulate at $\gamma(t_0)$ must split from the slice containing $\gamma(t_0)$.

Let $B_{\pi/8}(\gamma(t_0))$ be a $\pi/8$ - uniform ball for ∂D containing $\gamma(t_0)$, and note that some $\gamma(t_k) \in B_{\pi/8}(\gamma(t_0))$. But in $B_{\pi/8}(\gamma(t_o))$ we can apply Lemma 12, which would imply that $h(\gamma(t_0)) > 0$, a contradiction that stems from the assumption that $\gamma \cap \mathfrak{A}$ is not open. So, $\mathfrak{A} \cap \gamma$ is open in γ, and $z \in \mathfrak{A}$. $\square$

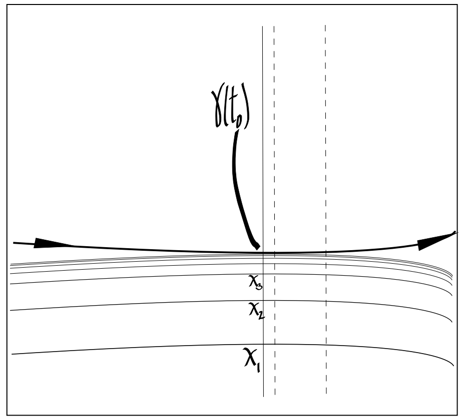

Figure 4: The Splitting point at $\gamma(t_0)$.

Lemma 13. *Let $\mathcal{D}$ be a uniformly planar domain. Then $\partial\mathcal{D}$ is locally connected, with respect to the topology of $\mathbb{R}^n$.*

Proof: By Proposition 4, if $\mathfrak{A}$ is not empty then it is all of $\partial\mathcal{D}$, because $\partial\mathcal{D}$ is path connected.

We now prove that it is impossible that $\mathfrak{A} = \partial\mathcal{D}$: Let x_0 be any point in $\partial\mathcal{D}$ ($= \mathfrak{A}$), $B_{\pi/8}$ a $\pi/8$- uniform ball containing x_0. Note that, in $B_{\pi/8}$, the intersections of $\ell(x_0)$ with $\partial\mathcal{D}$ are transversal. Let

$$x \in B_{\pi/8} \cap \overline{\mathcal{D}} \cap \ell(x_0)$$

If x is in the interior of $\mathcal{D}$, it is clear that it is contained in a segment $x \in I_x \subset \overline{\mathcal{D}} \cap \ell(x_0)$ with non-empty interior. If $x \in \partial\mathcal{D}$, then by transversality x is an endpoint of a segment $I_x \subset \overline{\mathcal{D}} \cap \ell(x_0)$ with non-empty interior. Hence, we can write

$$B_{\pi/8} \cap \overline{\mathcal{D}} \cap \ell(x_0) = \bigcup_{x \in B_{\pi/8} \cap \overline{\mathcal{D}} \cap \ell(x_0)} I_x$$

Where all segments I_x have non-empty interior. Thus, there exists a countablepp subfamily of line segments $\{I_j\}_{j \in \mathbb{N}}$ in the family $\{I_x\}_{x \in B_{\pi/8} \cap \overline{\mathcal{D}} \cap \ell(x_0)}$ such that

$$B_{\pi/8} \cap \overline{\mathcal{D}} \cap \ell(x_0) = \bigcup_{j \in \mathbb{N}} I_j$$

Taking the appropriate coordinates, each I_j can be written $I_j = [x_{2j}, x_{2j+1}] \subset \mathbb{R}$, where $x_i \in \partial\mathcal{D}$ and $x_i \neq x_j$ if $i \neq j$. If we assume, to get a contradiction, that $\mathfrak{A} = \partial\mathcal{D}$, then one has

$$B_{\pi/8} \cap \mathfrak{A} \cap \ell(x_0) = \bigcup_{j \in \mathbb{N}} \{x_{2j}\} \cup \{x_{2j+1}\} \tag{158}$$

(158) implies that the set $B_{\pi/8} \cap \mathfrak{A} \cap \ell(x_0)$ is countable. But one also has that each $x_i \in \mathfrak{A}$, and so x_i is an accumulation point for $B_{\pi/8}$ - slices which necessarily intersect $B_{\pi/8} \cap \partial\mathfrak{D} \cap \ell(x_0)$. Thus, the set $B_{\pi/8} \cap \mathfrak{A} \cap \ell(x_0)$ is a perfect set, and thus uncountable if non-empty, contradicting (158). Thus $\mathfrak{A}$ must be empty. $\square$

Lemma 14. *Let $\mathcal{D}$ be a uniformly planar domain in $\mathbb{R}^n$ such that $\partial\mathcal{D}$ is locally connected. Then $\mathcal{D}$ is compensated.*

Proof: It suffices to show that $\partial\mathcal{D}$ is compensated: Indeed, if either of x_0, y_0 is an interior point of $\mathcal{D}$, let ℓ be the line segment joining x_0, y_0 and let x_1, y_1 be the points in $\partial\mathcal{D} \cap \ell$ closest to x_0, y_0 respectively. Then, denoting the distance in $\mathcal{D}$ inherited from $\mathbb{R}^n$ by $d_{\mathcal{D}}$ and $C \geq 1$ the compensation constant for the boundary (as in (157)), we have:

$$d_{\mathcal{D}}(x_0, y_0) \leq |x_0 - x_1| + |y_1 + y_0| + d_{\mathcal{D}}(x_1, y_1)$$

$$\leq |x_0 - x_1| + |y_1 - y_0| + C|x_1 - y_1|$$

$$\leq C\Big(|x_0 - x_1| + |y_1 - y_0| + |x_1 - y_1|\Big)$$

$$\leq C|x_0 - y_0|$$

where in the last inequality we used that x_1, y_1 are in the straight line segment joining x_0, y_0.

To show that $\partial\mathcal{D}$ is compensated: Cover $\partial\mathcal{D}$ with $\frac{\pi}{4}$ -uniform balls $\{V_x\}_{x\in\partial D}$. Since $\partial\mathcal{D}$ is locally connected, we can take $\{U_x\}_{x\in\partial\mathcal{D}}$, $U_x \subset V_x$, such that the sets $\{U_x \cap \partial\mathcal{D}\}_{x\in\partial\mathcal{D}}$ are path connected. Using compactness of $\partial\mathcal{D}$ we now extract a finite subcover $\{U_i\}_1^M$. Let d be the maximum diameter of the balls in this cover.

Given $x, y \in \partial\mathcal{D}$, we construct a path γ in $\partial\mathcal{D}$ joining x and y such that length$(\gamma) \leq 2Md$. As $\partial\mathcal{D}$ is path connected, let $\tilde{\gamma} : [0,1] \to \partial\mathcal{D}$ be a C^1 curve with $\tilde{\gamma}(0) = x$ and $\tilde{\gamma}(1) = y$. We now create shortcuts to shorten $\tilde{\gamma}$ as follows: Let $\tilde{\gamma}(0) \in U_{i_0}$ and $\tau^0 = \sup\{t : \gamma(t) \in U_{i_0}\}$. As $U_{i_0} \cap \partial\mathcal{D}$ is path connected, and it is a $\pi/4$-uniform ball, the conditions of Proposition 3 are satisfied and thus there exists $\gamma_1 : [0,1] \to \partial\mathcal{D}$ joining $\tilde{\gamma}(0)$ and $\tilde{\gamma}(\tau^0)$ such that length$(\gamma_1) \leq 2d$. γ_1 is the first shortcut. We now repeat the same process starting at τ^0 instead of 0, and obtain a second shortcut γ_2. We repeat this process at most M times and we obtain a sequence of concatenated paths $\{\gamma_i\}_1^N$, $N \leq M$, with length$(\gamma_i) \leq 2d$. The concatenation $\gamma := (\ldots(\gamma_1 \bullet \gamma_2) \bullet \ldots \gamma_{N-1}) \bullet \gamma_N$ is a piecewise C^1 path in $\partial\mathcal{D}$ joining x and y and of length no greater than $2Md$. $\square$

CHAPTER X

COUNTEREXAMPLES FOR TENTATIVE GENERALIZATIONS OF THEOREM 2.

In this Chapter we provide counterexamples for tentative generalizations of Theorem 2: First, to non-uniformly planar immersed manifolds of $\mathbb{R}^n$ and then to uniformly planar manifolds in an infinite dimensional space.

The main idea for obtaining the counterexamples is the following: We create first a curve in $\mathbb{R}^n$ or in ℓ^2 where compensation is not satisfied, and then we construct domains in $\mathbb{R}^n$ or ℓ^2 modelled after the curve as follows: We recall that given an embedded submanifold M of a Hilbert space $\mathcal{H}$ (in our case $\mathbb{R}^n$ or ℓ^2), such that M is modelled after a proper subspace of $\mathcal{H}$, (in the finite dimensional case, that is $\dim M < \dim \mathcal{H}$), by the Implicit Function Theorem, there is an open set of $\mathcal{H}$ containing M, U, which we will call *tubular neighbourhood of* M with the following property: There is a diffeomorphism

$$E : NM \supset V \longrightarrow U$$

$$E(x, v) = x + v$$

where NM is the normal bundle of M, and V is a subset of NM of the form $V = \{(x, v), \|v\|_{\mathcal{H}} \leq \phi(x)\}$, $\phi(x)$ a positive continuous function that is less or equal to the injectivity radius of the normal bundle to M at x (see e.g. [39]).

Thus, using the coordinate chart given by the diffeomorphism E, we have that the set $\mathcal{T} \subset \mathcal{H}$ given by:

$$\mathcal{T} := \{E(y) : NM \ni y = (x, v), \|v\|_{\mathcal{H}} \leq \varphi(x)\}$$

where $\phi(x) > \varphi(x) : M \longrightarrow \mathbb{R}$ is a C^1 function on M, is a C^1 embedded manifold in $\mathcal{H}$ with boundary $\{E(z) : z = (x, v), \|v\|_{\mathcal{H}} = \varphi(x)\}$. The manifold with boundary $\mathcal{T}$ will be the counterexample.

10.1 Counterexamples for more general domains in finite dimension

We give examples of non-compensated domains which fail to satisfy only $\mathbf{C}_1$ (boundedness of $\mathcal{D}$) or $\mathbf{C}_3$ (existence of $B_\epsilon(x)$). It is very easy to give examples of non-compensated domains that fail to satisfy only $\mathbf{C}_2$ ($\partial\mathcal{D}$ is an immersed manifold).

Counterexample 1, to show that the condition of *uniformly* C^1 domain is necessary for Theorem 2: We present an example of a non compensated bounded domain in $\mathbb{R}^2$ with immersed C^1 boundary which is not uniformly planar (it satisfies conditions $\mathbf{C}_1$ and $\mathbf{C}_2$, but not $\mathbf{C}_3$). We suggest to the reader to look at Figure 5 below to better follow the easy construction of the domain $\mathcal{T}$ without getting lost in the necessary notation. To simplify notation, we denote $t_1 := \frac{2}{3\pi}$, and $t_2 := \frac{2}{3\pi} + \frac{3\pi}{2}$. Let:

$$\tilde{\gamma}(t) : [t_1, t_2] \to \mathbb{R}^2$$

be a parameterization of a 3/4 arc of a circle such that $\tilde{\gamma}(t_1) = (t_1, -1)$ has horizontal tangent line and $\tilde{\gamma}(t_2) = (t_1^3/10, -1 - t_1 - t_1^3/10)$ has a vertical tangent line. Consider the planar curve

$$\gamma(t) = \begin{cases} \left(t, \sin\left(\frac{1}{t}\right)\right), & t \in (0, t_1) \\ \tilde{\gamma}(t), & t \in [t_1, t_2] \\ \left(-t_1^3/10, \tilde{\gamma}(t_2) + t - t_2\right), & t \in (t_2, t_2 + 3) \end{cases}$$

Estimating the injectivity radius of the normal bundle of γ at t, $R(t)$ by the radius of curvature of γ at t, for the sinusoidal part of γ, $t \in (0, \frac{2}{3\pi})$, we obtain that $R(t) > |t|^3/10$. Hence, if we denote by $\hat{n}(t)$ a unit normal vector to γ at time t, the planar set:

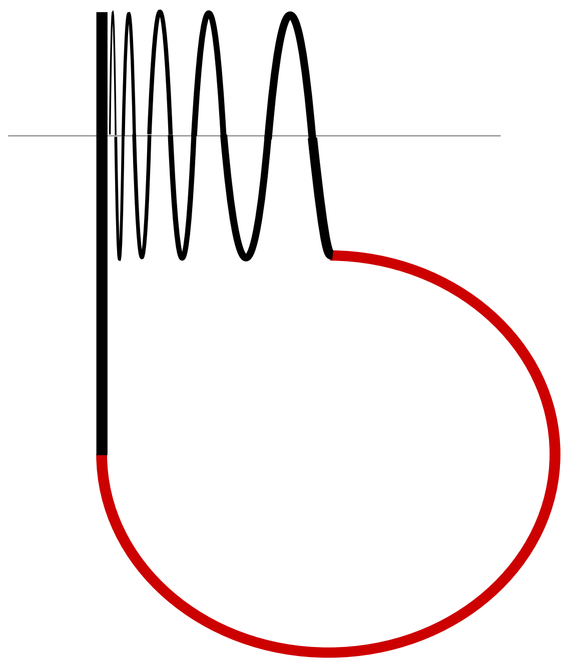

Figure 5: An illustration of $\mathcal{T}$ with the tubular neighbourhood of $\tilde{\gamma}$ highlighted.

$$\mathbb{R}^2 \supset \mathcal{T} = \begin{cases} x : x(t,\delta) = \gamma(t) + \delta\hat{n}(t), & |\delta| < |t|^3/10, \quad t \in (0, t_1) \\ x : x(t,\delta) = \gamma(t) + \delta\hat{n}(t), & |\delta| < t_1^3/10, \quad t \in [t_1, t_2 + 3] \end{cases}$$

is a bounded domain with boundary $\partial\mathcal{T}$ a C^1 immersed manifold. We observe that

$\partial\mathcal{T}$ is not uniformly planar, indeed, the point $(0,1)$ is not contained in any $\pi/4$-uniform ball: Any open ball containing $(0,1)$ contains a neighbourhood W of the sequence $x_k := \{(1/k\pi, 1)\}_{k\in\mathbb{N}}$, and one sees that the bundle of normal vectors to W cannot be translated to a single $\pi/4$- cone.

Finally we observe that the domain $\mathcal{T}$ is not compensated. Indeed for the sequence x_k introduced in the previous paragraph, one has

$$\|(0,1) - x_k\|_{\mathbb{R}^2} = 1/k\pi$$

for the Euclidean norm of $\mathbb{R}^2$, whereas for the distance $d_{\mathcal{T}}(\cdot,\cdot)$ on $\mathcal{T}$ induced from the Euclidean norm of $\mathbb{R}^2$ (defined as in (157)), one has $d_{\mathcal{T}}((0,1), x_k) > k$.

Counterexample 2: We give an example of a noncompensated domain $\mathcal{T}$ in $\mathbb{R}^2$ that satisfies conditions $\mathbf{C}_2$ and $\mathbf{C}_3$, but not the boundedness condition $\mathbf{C}_1$. The idea of the construction is to create parallel lines which we will connect by arcs. Two consecutive parallel lines can be drawn arbitrarily close if the arcs that connect them are sufficiently far away.

Let $\mathcal{C}_1$ be the quarter arc of circle $\{(\cos t, \sin t)\ t \in [3/2\pi, 2\pi]\}$, $\mathcal{C}_2$ be the half arc of circle $\{(\cos t, \sin t)\ t \in [0,\pi]\}$ and $\mathcal{C}_3^k$ be the quarter arc of circle $\{(1 + 2^{-k})(\cos t, \sin t)\ t \in [3/2\pi, 2\pi]\}$. Let $\tilde{\gamma}_k$ be the concatenation of these curves, so that $\tilde{\gamma}_k$ is a connected and smooth curve composed of three arcs of circle of radius greater or equal to 1, joining the points $(0,0)$ and $(0,-2^{-k})$ (see Figure 6 below). Let $\tilde{\gamma}_k : [0,1] \to \mathbb{R}$ be a parameterization of $\tilde{\gamma}_k$. Let $-\tilde{\gamma}_k$ be the curve obtained by reflecting $\tilde{\gamma}_k$ about the y axis.

Let $\gamma(t) : [0,\infty) \to \mathbb{R}^2$ be the non-compact planar curve parameterized recursively as follows:

$$
\gamma(t) = \begin{cases} (kt, 0) + \gamma(4k) \quad t \in [4k, 4k+1], k \in \mathbb{N} \\ \tilde{\gamma}(t - 4k - 1) + \gamma(4k+1) \quad , t \in [4k+1, 4k+2] \\ (-kt, 0) + \gamma(4k+2), \quad t \in [4k+2, 4k+3] \\ -\tilde{\gamma}(t - 4k - 3) + \gamma(4k+3), \quad t \in [4k+3, 5k] \end{cases}
$$

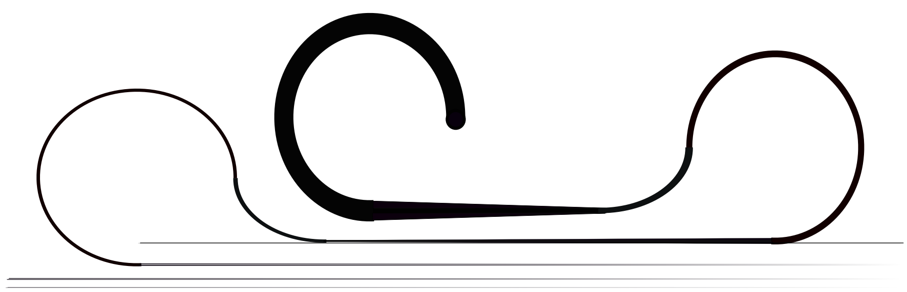

Figure 6: The first steps in the construction of $\mathcal{T}$.

Graphically, γ consists of a family of segments accumulating at the x axis, and connected consecutively by curves that have radius of curvature greater or equal to 1. Hence the local radius of injectivity of the normal bundle of γ at any point is greater or equal to 1. Let $\varphi_k(t) : [0,1] \to \mathbb{R}$ be a C^1 bump function such that $\varphi_k(0) = 3^{-1}2^{-k-1}$, $\varphi_k'(0) = 0$, $\varphi_k(1) = 3^{-1}2^{-k-2}$, $\varphi_k'(1) = 0$. Thus, the planar set $\mathcal{T}$ given by

$$
\mathcal{T} = \begin{cases} x = \gamma(t) + (0, y) \quad y \in [-\varphi_k(t), \varphi_k(t)], \quad t \in [4k, 4k+1], k \in \mathbb{N} \\ x : d(x, \gamma(t)) \leq 3^{-1}2^{-k-2} \quad , t \in [4k+1, 4k+2] \\ x = \gamma(t) + (0, y), y \in [-3^{-1}2^{-k-2}, 3^{-1}2^{-k-2}], \quad t \in [4k+2, 4k+3] \\ x : d(x, \gamma(t)) < 3^{-1}2^{-k-2}, \quad t \in [4k+3, 5k] \end{cases}
$$

is a uniformly planar immersed submanifold with boundary in $\mathbb{R}^2$, but it is not compensated, because it contains the sequence $x_k = \{(0, 2^{-k})\}_{k \in \mathbb{N}}$, and one has $\|x_k, x_{k+1}\|_{\mathbb{R}^2} < 2^{-k-1}$, whereas $d_{\mathcal{T}}(x_k, x_{k+1}) > 1$, where $d_{\mathcal{T}}$ is the riemannian distance induced on $\mathcal{T}$ from $\mathbb{R}^2$.

10.2 Counterexample for uniformly planar manifolds in infinite dimension.

In this Section we give an example of a non-compensated uniformly planar domain in ℓ^2, the Hilbert space of square summable sequences of real numbers.

Let $\{\tilde{\gamma}_k\}_{k\in\mathbb{N}}$ be the planar curve given parametrically by:

$$\tilde{\gamma}_k(t) = \begin{cases} (0,t), \ t \in (1/2, 2) \\ (-5\cos(t-2)-1, 5\sin(t-2)+2), \ t \in (2, 2+\pi) \\ (5, -t+2), \quad t \in (\pi+2, \pi+7) \\ (5\cos(t-\pi-7), -5\sin(t-\pi-7)-2), \ t \in (\pi+7, 2\pi+7) \\ (0, t-2\pi-5), \ t \in (2\pi+7, 2\pi+8-100^{-1}-k^{-1}) \end{cases}$$

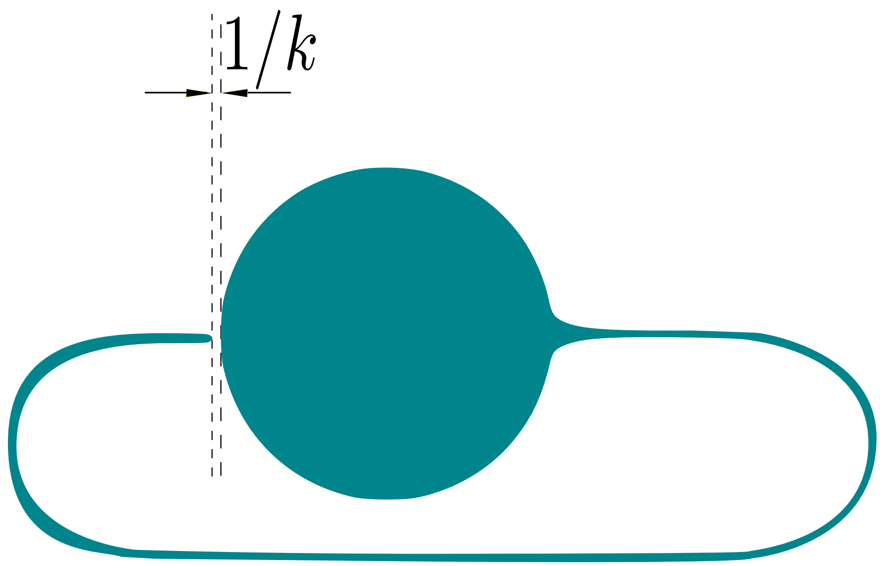

Figure 7: $\mathcal{T} \cap E_{2k}$ is homeomorphic to this planar region.

Note that the endpoint of $\tilde{\gamma}_k$ close to the south pole of the unit ball depends on k, becoming closer to the south pole as k tends to infinity. Graphically, $\tilde{\gamma}_k$ is composed of two half- circles and three lines, properly connected as to obtain a C^1 curve that starts inside the unit ball, closer to the north pole and ends near the south pole (see

Figure 7). From the family $\{\gamma_k\}$ we construct an isometric family of curves in ℓ^2, the Hilbert space of square- summable real sequences, with the Euclidean distance induced from its scalar product, as follows. We introduce the following notation: Let π_i be the projection operator of ℓ^2 onto the i-th coordinate. For $k \in \mathbb{N}$, define E_{2k}, the following two- dimensional subspaces of ℓ^2:

$$E_{2k} = \{x \in \ell^2 : \pi_i(x) = 0,\ \forall i \neq 2k, 2k+1\}$$

For $k \in \mathbb{N}$, define $\tilde{E}_k$ as the subspace of ℓ^2 given by:

$$\tilde{E}_k = \{x \in \ell^2 : \pi_k = 0\}$$

Let $\tilde{\pi}_k : \ell^2 \to \ell^2$ be the orthogonal projection operator of ℓ^2 onto $\tilde{E}_k$.

Then we denote the following planar curves embedded in ℓ^2:

$$\ell^2 \ni \gamma_k(t) = \left\{ x \in E_{2k} : (\pi_{2k}(x), \pi_{2k+1}(x)) = \frac{1}{2^k}\tilde{\gamma}_k(t) \right\}$$

If we denote $\gamma_k,\ \tilde{\gamma}_k$ the sets associated to the curves, one sees that, for $k \neq k'$

$$\mathrm{dist}(\gamma_k, \gamma_{k'}) = \frac{\sqrt{2}}{2}$$

Where dist is the infimum of the distances of points in the two sets. Observe that the union of these sets is contained in a ball of finite radius in ℓ^2.

As in the previous counterexamples, we use a tubular neighbourhood of the family of curves γ_k to construct a counterexample to Theorem 2. Observe that the radius of injectivity of the normal bundle to each γ_k is greater or equal to 1 (it is greater or equal to the radius of curvature of the half- circles) . As the distance between any two curves in $\{\gamma_k\}_{k\in\mathbb{N}}$ is greater or equal than $\sqrt{2}/2$, we see that the radius of injectivity of the normal bundle of the set $\cup_k \gamma_k$ is greater or equal to $\sqrt{2}/4$. Thus the set

$$\mathcal{T} = \{x \in \ell^2 : \mathrm{dist}(x, \gamma_k) \leq 100^{-1}, \text{for some k} \in \mathbb{N}\}$$

is a C^1 manifold with boundary in ℓ^2. We will now attach the unit ball of ℓ^2 to this tubular neighbourhood and smooth out the intersection as follows: Let $C^1(\mathbb{R}) \ni \varphi : \mathbb{R} \to \mathbb{R}$ be a bump function such that

$$\varphi(0.9) = \sqrt{1-(0.9)^2}, \qquad \varphi'(0.9) = \frac{-0.9}{\sqrt{1-(0.9)^2}}$$
$$\varphi'(2) = 0, \qquad \varphi(2) = 100^{-1}$$

Such a bump function is easy to create using the function $e^{-1/x}$, but to keep the notation simple we just postulate that it exists. Note that φ is tangent to the unit circle in $\mathbb{R}^2$ at the point $(0.9, \sqrt{1-(0.9)^2})$, and is tangent to the boundary of the tubular neighbourhood of the x axis of radius 100^{-1}. Thus the set of ℓ^2 given by:

$$\mathcal{U} = \{\ell^2 \ni x : \pi_{2k+1} = t,\ \|\tilde{\pi}_{2k+1}(x)\|_{\ell^2} \leq \varphi(t), \quad t \in [0.9, 2],\ k \in \mathbb{N}\}$$

is an embedded manifold with boundary in ℓ^2, modelled after ℓ^2. This manifold is composed of countably many connected components which are tubular neighbourhoods of the curves $\{\gamma_k\}$ and are tangent to the unit ball of ℓ^2, B_1, and to $\mathcal{T}$. Hence the set $\mathcal{M} = \mathcal{T} \cup \mathcal{U} \cup B_1$ is an embedded C^1 manifold with boundary modelled after ℓ^2, which furthermore is uniformly flat. In the Figure bellow we have drawn the projection of $\mathcal{M}$ onto the subspace E_{2k}.

However, $\mathcal{M}$ is not compensated. Indeed for the families of points $\{y_j\} \in B_1$, $\{x_j\}_{j\in\mathbb{N}} \subset \mathcal{M}$ given by the condition:

$$E_{2j} \ni x_j\ \pi_{2j}(x_j) = 0, \quad \pi_{2j+1}(x_j) = -1 - j^{-1}$$

$$E_{2j} \ni y_j,\ \pi_{2j}(y_j) = 0,\ \pi_{2j+1}(y_j) = -1$$

We have $\|x_j - y_j\|_{\ell^2} = j^{-1}$, but $\|x_j - y_j\|_{\mathcal{M}} > 1$, where $\|\ \|_{\mathcal{M}}$ is the distance induced on $\mathcal{M}$ by the Hilbert norm of ℓ^2 (defined as in (157)). Thus $\mathcal{M}$ is not compensated.

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